A Proof of the Reconstruction Conjecture

Faust, Johann *
University of Washington
Seattle, WA, USA

Mephistopheles†
Offices of the Infernal Legions
Malebolge, Eighth Circle, Hell

September 8, 2024

Abstract

The reconstruction conjecture, proposed by Kelly and Ulam, asks whether a graph is uniquely determined by its multiset of proper induced subgraphs. Despite substantial effort, over the 70 years since its proposal only modest progress has been made towards resolving the problem. In this work, we prove the statement in full generality. The techniques are novel, involving the sacrifice of one author's soul in exchange for earthly and divine knowledge from beyond the grave.

1 Introduction

[**JF:** So we're actually doing this on Overleaf? I thought you were joking.]

[**M:** hey, nothing wrong with overleaf. you know those "server disconnected" syncing errors? i know the guy who came up with that one – big stuff. anyway yea, all that matters is that you do this of your own free will and understand what youre giving up yada yada. go ahead and start writing.]

1.1 Background and History

In 2012, the first author learned about the problem via personal communication from Gretchen Anderson. A January blizzard was in the process of burying Ann Arbor in 14 inches of snow – it was perhaps 5 in the afternoon, but the darkness outside made it feel surreally late. He and Anderson were sitting together on the tiled floor of her tiny kitchen, sipping hot chocolate as they played Go Fish and tried to forget about their problem sets.

When they'd first broken out the deck, they'd been full of energy. The first author had laughed at Anderson as she, mummified by a mass of fuzzy blankets, struggled to drink from her mug or hold her cards without shifting from her comfortable sitting position. Every time Anderson told the first author to "go fish", she would supplement with some snide remark or boastful taunt. But eventually the game had settled into a state of comfortable silence, half-yawned requests for fives and jacks accompanied only by the sound of cards being rearranged and the distant banshee cries of the wind outside.

This silence was broken as Anderson, shifting her leg, knocked over her mug, producing a small pool of brown liquid that slowly seeped into the cards in front of her. The first author (who was losing the round badly) leapt to his feet, rescuing the dampened cards and accusing her with mock indigence of sabotaging the game. Once the floor had been wiped and the cards – cleaned as best they could but still slightly sticky – returned to a single pile, he proclaimed cheerfully "well, I guess there's no way now to tell who was winning!"

Anderson laughed and made a face at him. "I bet we've got enough information left to be pretty sure." Her eyes narrowed into the sly look she always gave when trying to segue an innocent conversation into math. "You know... it's kinda like the reconstruction conjecture."


Rolling his eyes, the first author took the bait. "Ok, what's the reconstruction conjecture?"

The instant he responded, Anderson eagerly tossed aside her shroud of blankets, and began to write with a pen on her cocoa-stained napkin (see Figure 1). Smiling and shaking his head in a mixture of amusement and false exasperation, the first author scooched close, watching over her shoulder as she explained.
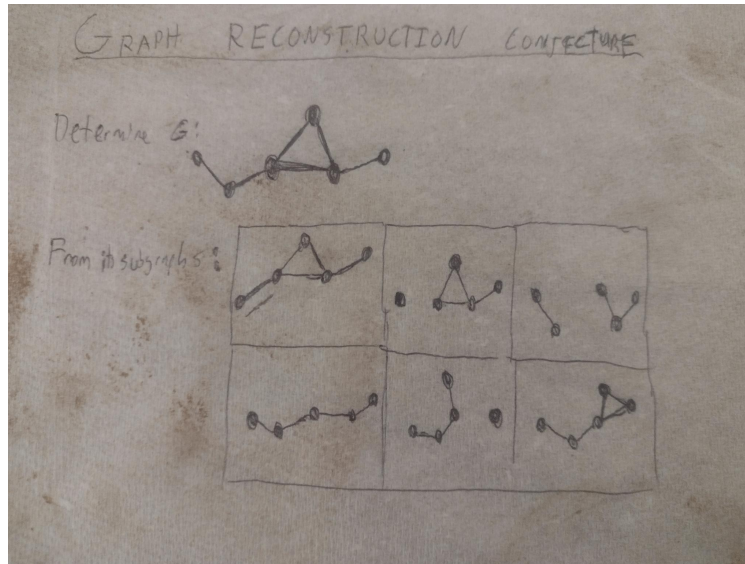
Figure 1: The original napkin Anderson wrote on that evening.

"Suppose I have a graph, I throw away one of its vertices, and I show you what's left. Can you figure out what the original was?"

She drew an example on the napkin. The first author looked at her quizzically – he had no idea where she was going with this.

"Of course not – it's like the cards, the information is just gone. But what if I showed you *all* the different ways to delete a vertex? Maybe, even though they're each missing something, if you stare hard enough at the fragments that remain there's some way of piecing back together the original."

"...is there?"

She raised her eyebrows and paused for a moment, turning to look at him with a barely-suppressed grin. The first author held his breath involuntarily, staring into her chocolate-brown eyes, as he waited what felt like an eternity for the answer. Her excitement was always contagious.

"Nobody knows."

1.2   Previous Work

In 2013, Anderson and Faust wrote a sprawling work analyzing reconstruction of **_perfect graphs_** [AF13]. The paper grew out of a year's worth of late nights in Anderson's kitchen, where she and Faust convened every evening for a graph theoretic witches' sabbath. Anderson would sit cross-legged on the counter, absent-mindedly picking fluff from her thick wool sweater, as they went back and forth, excitedly proposing new ideas and finding flaws in old ones, leaving faint smudges of red and blue and black and purple on a whiteboard too old to ever fully erase. In the end, it turned out that most of their results had already been shown, far more succinctly, in Section 4 of a 20-year-old journal article – so they never tried to publish anywhere. Which is probably for the best, anyway, because their write-up was a total mess, full of photos of hand-sketched figures and thickets of indecipherable lemma statements. Faust

was devastated that they'd been wasting their efforts re-deriving already known results. But even after Anderson had read those older proofs, for all of their classical elegance, she continued to maintain that theirs had been the more beautiful.

In 2015, Anderson and Faust, now working alongside their mutual advisor at UW, managed to publish their first paper. They defined a class of *joyful* graphs, and proved that all such graphs require a large number of subgraphs to reconstruct [AFK15]. Although the result was somewhat niche, the paper was written meticulously, full of wheels and sunflowers and stars that Faust had spent hours drawing at his computer while Anderson perched behind him, providing unsolicited commentary and trying to distract him with a card trick she could never quite get right. The day they got the acceptance notice, they went back to their apartment and, holding mugs of hot chocolate to the air, swore a solemn oath of battle: one day, together, they would defeat the reconstruction conjecture completely. And over the next couple years, though their victories won only modest ground, they continued to fight this battle side-by-side.

In 2016, Anderson and Faust identified a statement about order ideals that would imply that all posets are reconstructible [AF16]. The following year, they finished a sweeping 300-page tour-de-force, bounding the behaviour of reconstruction numbers in graphs formed from joins or unions of two smaller graphs [AF17].

In 2018, Faust and Faust studied reconstruction in the unitary Cayley graphs of rings [FF18a]. Later that year, they showed that it's always possible to reconstruct a graph if it's known to have a perfect matching [FF18b]. In 2019, in a sequence of surprising works, they showed that by searching for **stable sets** in a so-called **happy family** of subgraphs, the minimal set of proper subgraphs needed for reconstruction could be found in quasi-polynomial time [FF19a; FF19b; FF19c].

When, in 2023, Faust published his paper on reconstructing graphs with large holes [Fau23a], the graph theory community was relieved to see him picking himself up again. But as he continued to churn out a furious stream of incomprehensibly dense and error-filled results [Fau23b; Fau24a], survey papers that read more like the garbled ranting of a fever patient [Fau24b; Fau24c], and a frankly impressive number of flailing attempts to prove the reconstruction conjecture in a single stroke [Fau24d; Fau24e], this relief quickly turned to concern. His old advisor wrote to him "Please Johann, you have to let go. She wouldn't want to see you like this" [Kel24]. He never responded, but he thought to himself: "she would have wanted us to **know**".

1.3   This Paper

Three hours ago, the first author was walking back alone to his hotel from the day's session of the second annual Gretchen Faust Memorial Conference. Kela had given a beautiful talk in tribute, and as he walked the first author felt himself once again soaked in a sea of ill-defined emotions, staining the corners of his mind a dark brown until the memories seemed all to bleed into each other. He longed desperately for a way to cut through all the fuzziness that coated his thoughts, to scrub away the messy residues and see clearly enough to finally *understand*. But all he had was the same old mind he'd always had – locked in the same loops of guessing and second-guessing, going through the whole deck twice every time he tried to find the card he was thinking of.

"What would she think, if she could see me now?" he thought, in a wave of wretched self-pity. "How pathetic must I look, losing the same battle again and again, to the problem we swore to conquer together?" He stopped, proclaiming aloud to the empty street that he would give *anything* to have the answer – and something dark within him knew that he meant it.

As he continued homewards, lost in thought, he failed to notice that he had begun to be followed. At the sound of his outcry, a small, collarless black poodle had stepped gingerly from behind a dumpster – it now slunk down the street, peering into gutters and rubbing up against telephone poles, but always remaining about ten feet behind him. When he turned down a side street towards his hotel, it turned too. It wasn't until he'd walked into the lobby and stepped into the elevator that he finally looked down by his feet and came face to face with the second author. As the elevator rose, the first author watched the figure of the dog twist itself into the figure of a man, tall and neatly dressed, with a faint smirk on his face.

"I've come to make a deal."

2   Preliminaries

Definition 1. We call a subgraph induced by deleting a single vertex from a graph G a ***card***. The set of all cards forms the ***deck*** of G.

Definition 2. The ***soul*** is the essential structure of a human being. Without the soul, all that's left are the disconnected components – individual thoughts, memories, hopes, and experiences – with no underlying network to hold them together. It is unknown whether this structure, once lost, can ever be found again.

"For the low low price of one soul, I can offer you whatever strikes your fancy. Do you want your own personal guide to the pleasures and novelties of this world? I can take you to hell and back – and then some."

"I want her back."

"Sorry, no can do on the resurrections. We had an incident with some Helen of Troy stuff a while back, and it's been a hard no from the boss ever since. Anything else though – fame, power, a giant hamster wheel and a giant hamster to go with it, you name it. There'll be hell to pay – and we've got deep pockets."

"Then I want to know. I want to hear her prove reconstruction."

Definition 3. The ***reconstruction number*** of a graph is the number of cards needed to recover it after it's lost. The reconstruction conjecture holds that the reconstruction number is always defined.

Definition 4. The ***devil*** is the purest form of evil; the embodiment of all things hollow and empty. What he takes he never gives back.

"Do you understand what it will cost you?"

"I understand."

3   Methods

[**M:** ok, all youve gotta do now is prove that you are truly willing to make this deal. once we've got that done, i can contact your missus upstairs and get you that proof. ive gotta say, not the most exciting satanic pact ive ever seen – but hey, its your soul.]

[**M:** come now, lets get typing. you know what they say about idle hands.]

Lemma 1. The reconstruction number can be as high as $\frac{2(n-1)}{3}$. It's believed that this is tight – that the answer to the question "how much do you have to lose before there's no way back?" is 1/3. But all we know is the upper bound.

Proof. Let $r = \frac{(n-1)}{3}$, and consider the graphs $K_r \cup K_r \cup K_{r+1}$ and $K_{r-1} \cup K_{r+1} \cup K_{r+1}$. In either case, the card $K_{r-1} \cup K_r \cup K_{r+1}$ appears at least $2r$ times in the deck.   □

Lemma 2. The following are equivalent:
- The clumpy dregs at the bottom of a really rich cup of hot chocolate, when there's too much cocoa powder to dissolve and the liquid gradually becomes so thick and syrupy that you have to hold your head back, tilt the mug upside down, and wait for it to ooze down the sides.
- The soft warmth of an old alpaca wool sweater against your neck, its fuzzy surface dashed with the remnants of nicks from dry erase markers that never washed out, and a faint dusty smell like a UMich lecture hall.
- Heartbreak.

Proof. The night we swore our battle oath, she stayed uncharacteristically quiet throughout the festivities. She laughed and cheered along as I declared our sacred quest, but it sounded forced, as if her

attention was focused elsewhere, listening to the wind outside our window.

When, eventually, the celebration had subsided, we remained for some time wordlessly side-by-side. She leaned against the counter, staring fixedly ahead at a point on the wall by the refrigerator, slowly curling and uncurling her toes. I absent-mindedly washed the dishes, scrubbing the last traces of chocolate residue from the bottom of a mug with a paper towel. After what felt like an impossibly long time, she broke the silence.

"The other news we were waiting for came back today, too. It doesn't look good."

I felt suddenly very cold. My arms and legs shivered slightly, and without thinking I slumped down to the kitchen floor, my knees bent and my back pressed against the sink cabinet. She sat down beside me, turning my back gently away from the cabinet door and wrapping her arms around my neck. Her voice was almost inaudible.

"They say I have about five years in expectation, but the variance is large – we could easily have ten, if we're lucky."

All I could get out was, stupidly, "what do we do now?"

She gave a faint smile. "The same as we were always going to do. We huddle up through the winter. We play Go Fish. We prove some cool theorems."

"But what does it all mean, now that we already know how it ends?"

"We always knew how it ends."

<div align="right">□</div>

Proposition 1. I, Johann Faust, do willingly and without coercion surrender ownership of my soul to the demon Mephistopheles.

Proof. The proof follows from Lemma 1 and Lemma 2. Far, far more than 1/3 is gone. I have no use for what remains; you are free to take it. Just let her show me the end of the fight we couldn't win in time. □

4   Main Results

Theorem 1. All graphs are reconstructible.

Proof. [**GF:** The proof is left to the reader as an exercise.

The battle is worth it, Johann.] □

References

[AF13]   Gretchen Anderson and Johann Faust. "Reconstructibility Problems for Perfect Graphs". 2013.

[AFK15]   Gretchen Anderson, Johann Faust, and Easton Kela. "Joyful Graphs with Large Reconstruction Numbers". In: *Electronic Journal of Graph Theory* (Jan. 2015).

[AF16]   Gretchen Anderson and Johann Faust. "Towards the Ideal Conjecture in Poset Reconstruction". In: *Enumerative Combinatorics* (Mar. 2016).

[AF17]   Gretchen Anderson and Johann Faust. "Reconstructing Joins, Unions, and Products". In: *Panels of Mathematics* (Dec. 2017).

[FF18a]   Gretchen Faust and Johann Faust. "Ally Reconstruction Numbers for Unitary Cayley Graphs of Rings". In: *Journal of Combinatorics Theory* (Apr. 2018).

[FF18b]   Gretchen Faust and Johann Faust. "Class Reconstruction in Graphs with Perfect Matchings". In: *Graphica* (Nov. 2018).

[FF19a]   Gretchen Faust and Johann Faust. "Optimal Subdecks aren't Hard Unless PH Collapses". In: *Symposium To Advance Learning Knowledge*. Feb. 2019.

[FF19b]   Gretchen Faust and Johann Faust. "Algorithms for the Optimal Subdeck Problem". In: *Computational Optimization and Logical Algorithms*. June 2019.

[FF19c]   Gretchen Faust and Johann Faust. "Happy Families and Stable Sets: New Techniques for Optimal Subdecks". In: *Finite Optimization Xhibition*. Oct. 2019.

[Fau23a]   Johann Faust. *Barriers to Edge Reconstruction in Graphs with Large Holes*. Oct. 2023. arXiv: 12203.02773 [math.GT].

[Fau23b]   Johann Faust. "Polynomial Reconstruction of Disconnected and Degenerate Graphs". In: *Boulder And Denver Journal Of Underappreciated Research, Novels, And Life* (Dec. 2023).

[Fau24a]   Johann Faust. *No Time For Titles*. Feb. 2024. arXiv: 12203.03297 [math.GT].

[Fau24b]   Johann Faust. *Math is Cruel: Why We Will Never Solve Reconstruction*. Jan. 2024. arXiv: 12203.02993 [math.GT].

[Fau24c]   Johann Faust. *Forgive Me Gretchen, I Can't Do It*. Feb. 2024. arXiv: 12203.03023 [math.GT].

[Fau24d]   Johann Faust. *God is Mocking Us*. Mar. 2024. arXiv: 12203.03442 [math.GT].

[Fau24e]   Johann Faust. *40 Failed Proofs of the Reconstruction Conjecture*. Apr. 2024. arXiv: 12203.03565 [math.GT].

[Kel24]   Easton Kela. Personal communication. Mar. 2024.