What You Can’t Do With Math

A rather disjointed introduction to a couple impossibility theorems and stuff

Quickly introducing myself and the class

Me:

- I’m Nathan
- Freshman at MIT studying math
- From Boston area

The class:

- 3 stories; not particularly related to each other tbh
- All about things math shows us are impossible
- Vague outlines of the justifications, but no formal proofs unless we have extra time
- Will try not to assume much prior knowledge; if you’re confused tell me!

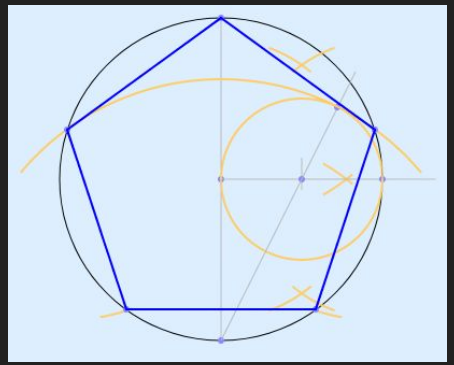
What You Can’t Do With Math #1: Drawing

Compass/Straightedge Constructions: The Problem

- Some ancient Greek people were pretty into drawing shapes

- Wanted systematic way of drawing shapes exactly right

- Can we generalize a minimal set of tools?

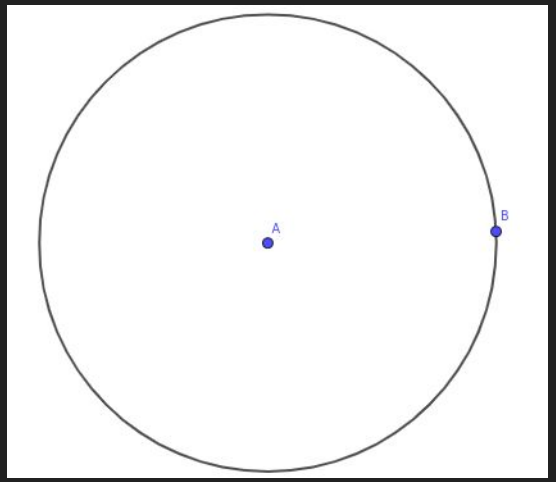


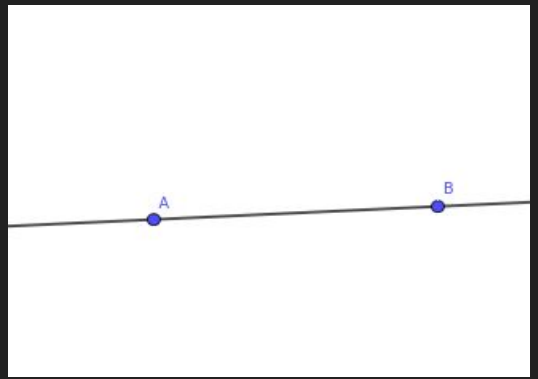
The Rules of the Game

- You have two tools: a compass and a straightedge

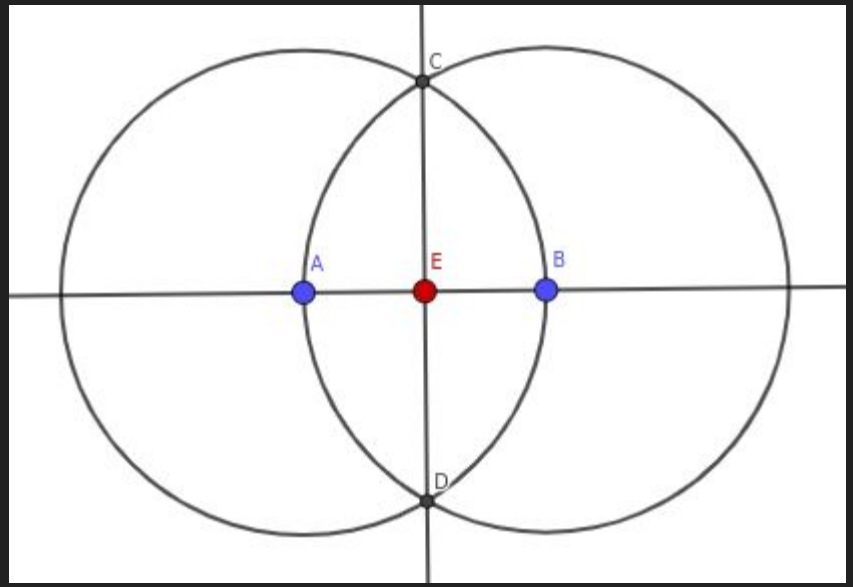
- Compass: given 2 points, draws a circle with one as the center and one on the circle

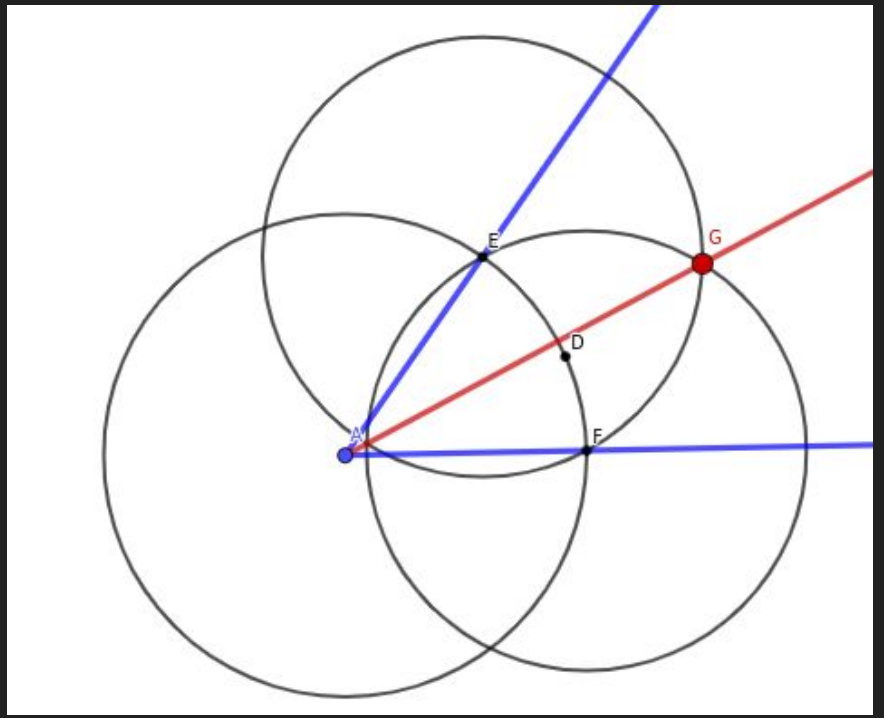
- Straightedge: given any 2 points, draws the line both lie on



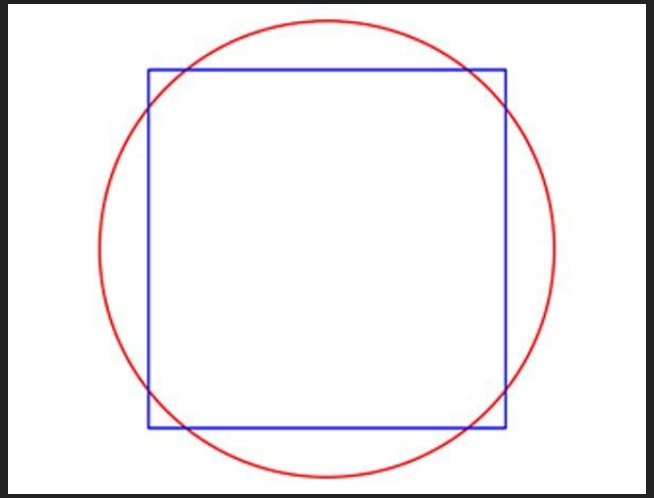


Actually quite powerful!

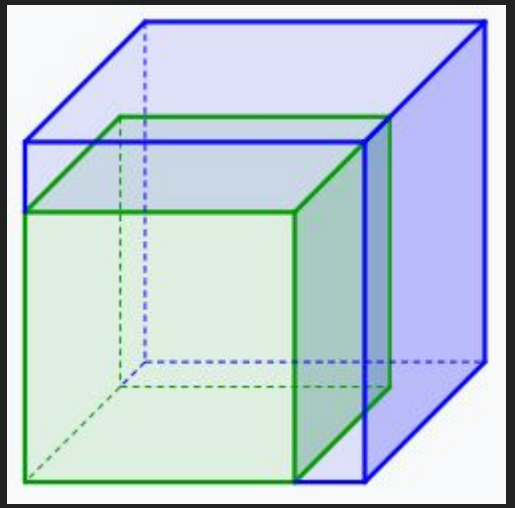




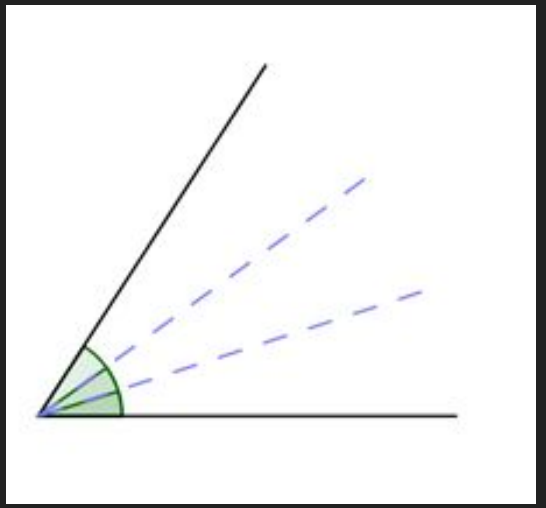
Some things are tricky though



Squaring the circle

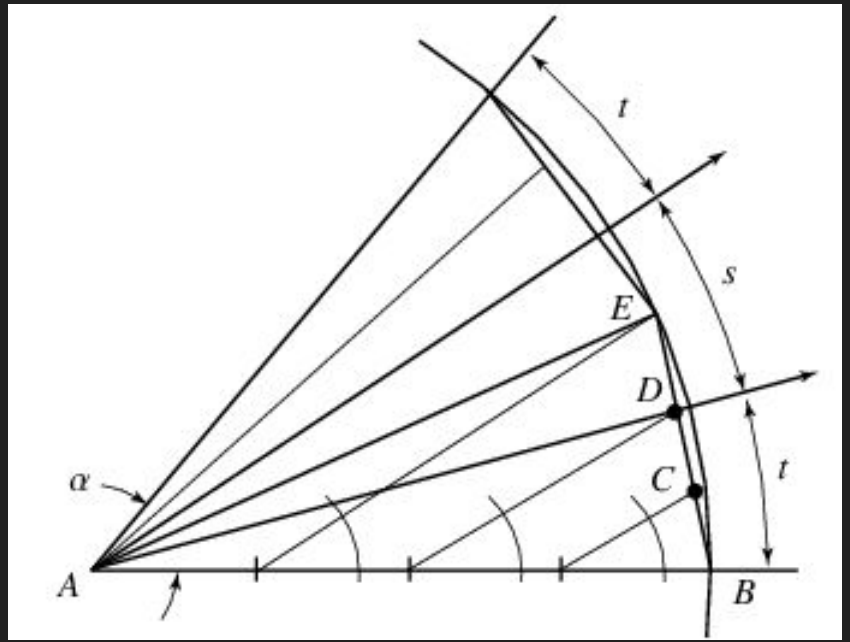


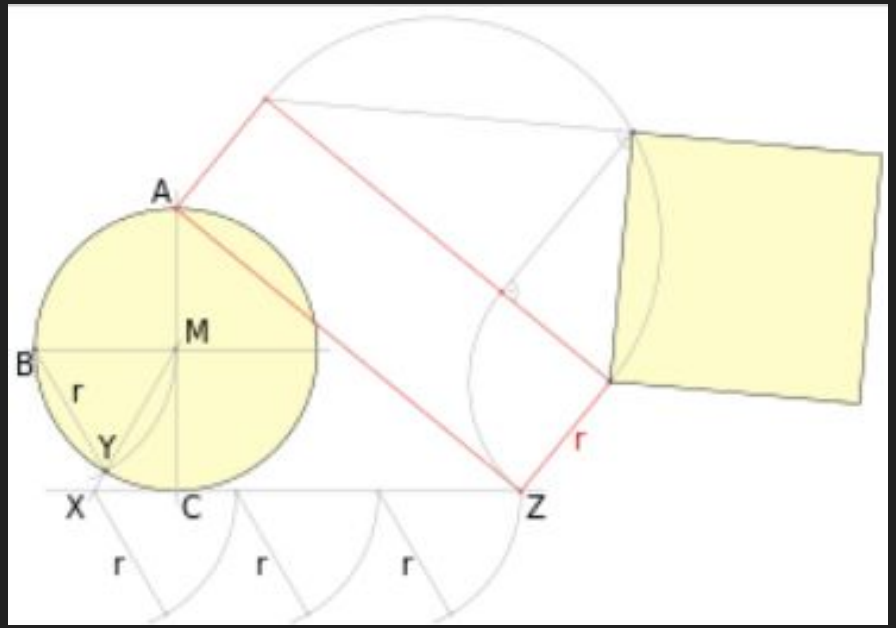
Doubling the cube



Trisecting an angle

People tried really hard, and got pretty close!





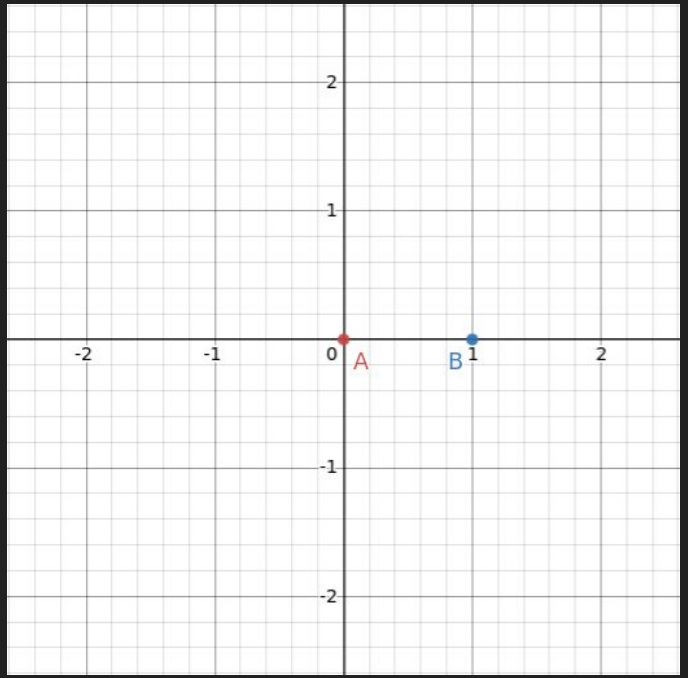
It turns out all 3 of these are impossible to do!!!!

Why? More formal statement of the problem

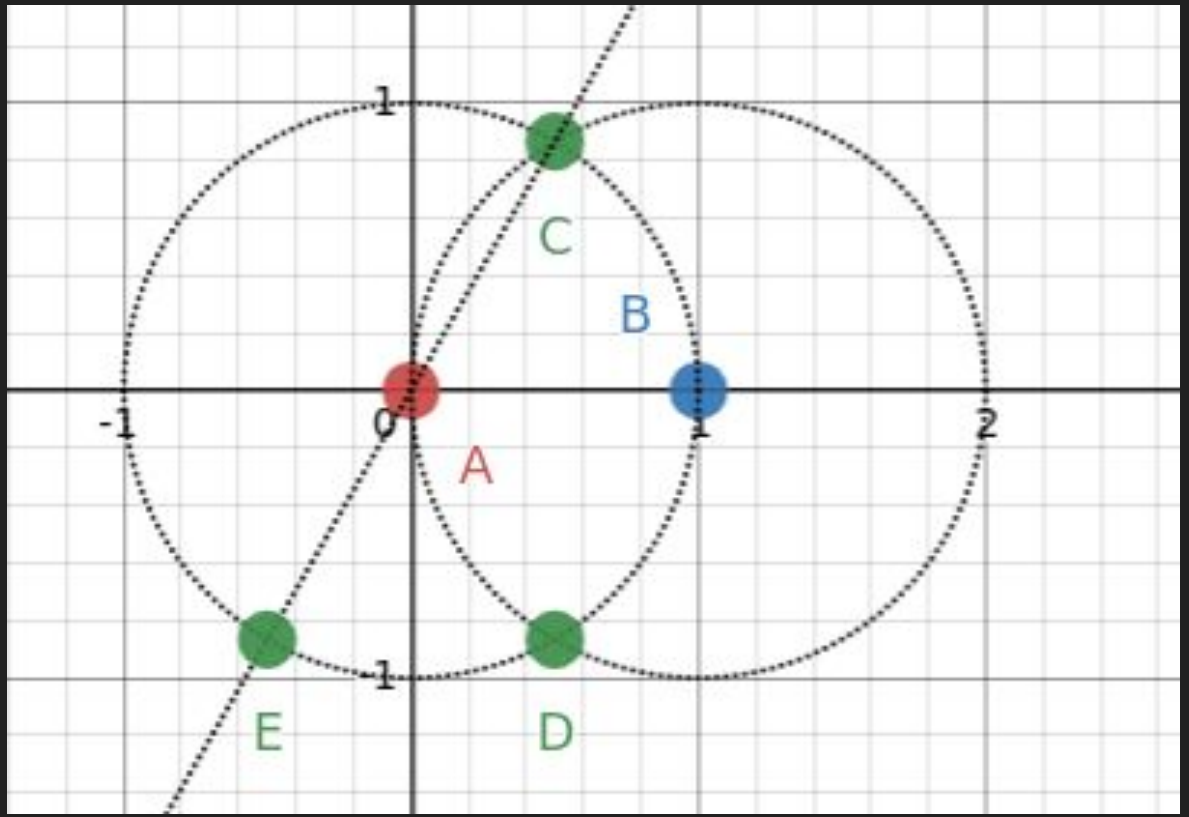
- Start with 2 points distance 1 apart

- What points is it possible to generate from these?

- This is a subset of the points in the plane; is it all of them?



If we call the set of points in the plane we can construct geometrically *S*, this picture shows that (1/2 , √3/2), (1/2, √3/2) and (-1/2, -√3/2) all belong to *S*



Converting geometric ideas to algebra

- We call a number “constructible” if it can be written as an expression using only { 1 , + , - , * , / , √ }

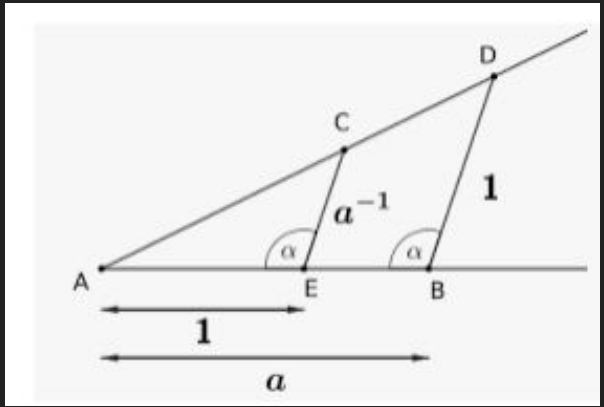
- So, for example, -(√(1+1+1))/(1+1) is a constructible number

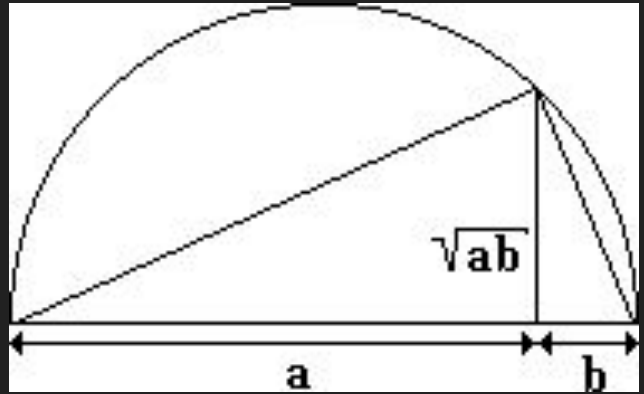
- Not all numbers are constructible! See π, $\sqrt[3]{2}$, etc.

- In math words, this is an example of a “Euclidean field”

Applying algebraic ideas back to geometry

- It turns out a point (a,b) is in *S* (that is to say, it can be constructed with a straightedge and compass starting from (0,0) and (0,1)) **if and only if** both a and b are constructible





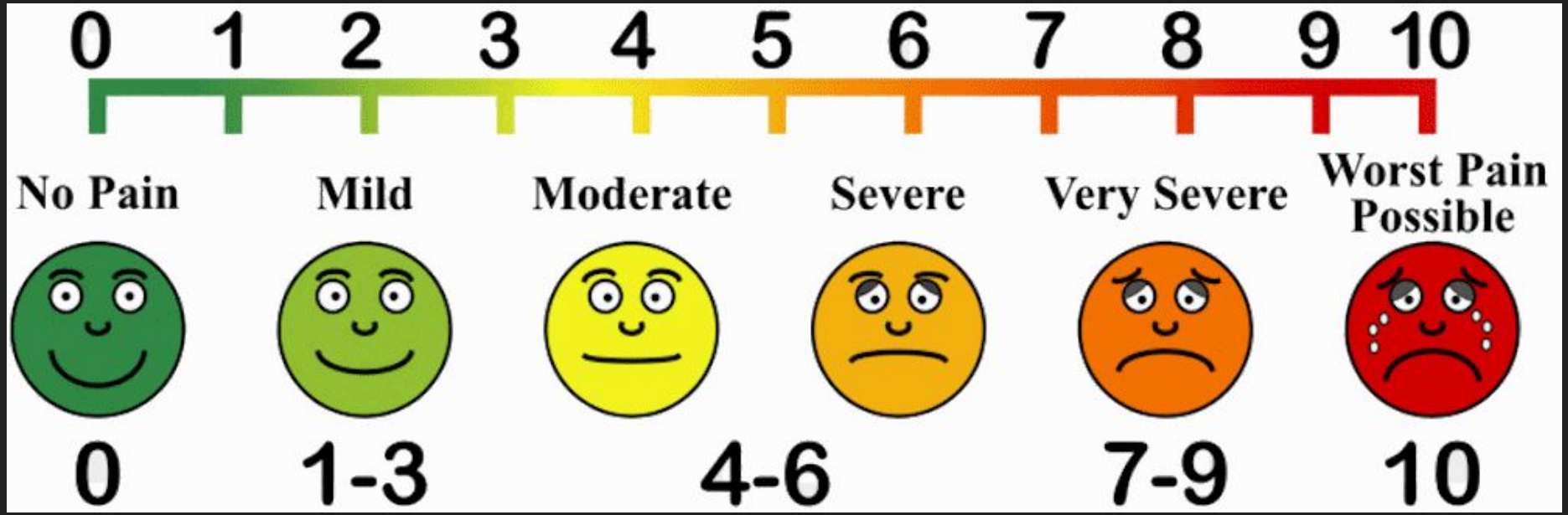
Proofs of impossible constructions

- Squaring the circle: requires constructing a line of length $\sqrt{\pi}$, but we showed π was not constructible!

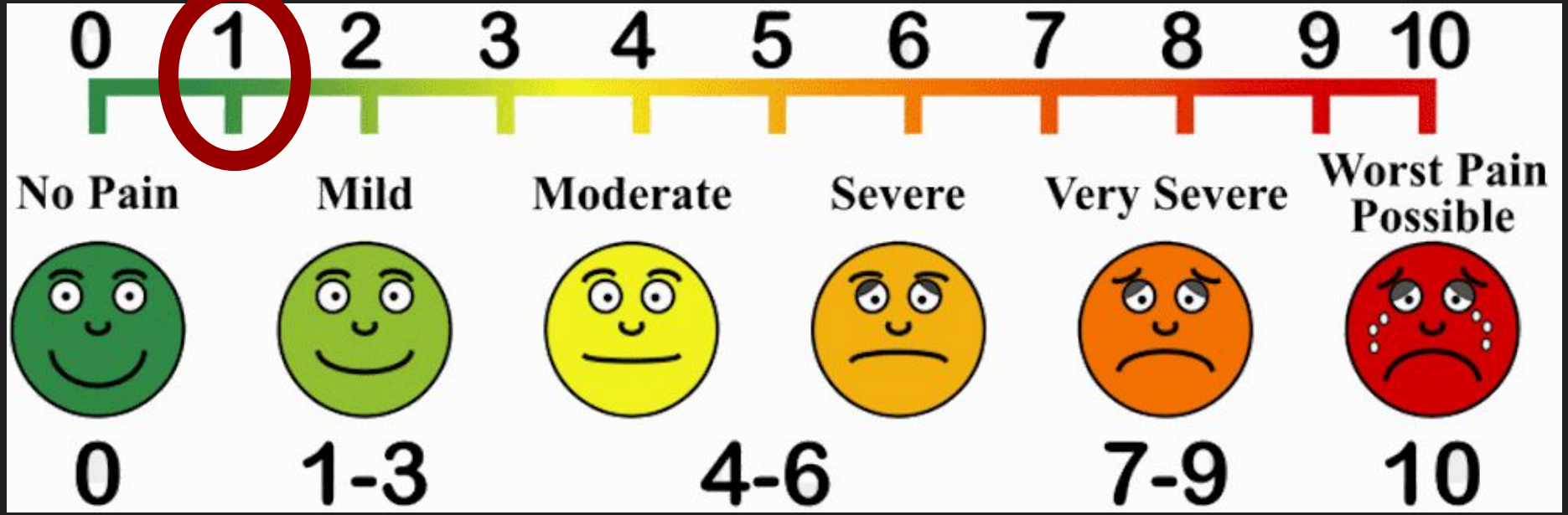
- Doubling the cube: side length scales by $\sqrt[3]{2}$; also not constructible

- Trisecting an angle: a 60° angle is constructible, since $\cos(60°) = \frac{1}{2}$ and $\sin(60°) = \sqrt{3}/2$ are both constructible numbers, but $\cos(20°)$ is not constructible (I won't prove it but it can be done)

How sad does this make you?



How sad does this make you?



What You Can’t Do With Math #2: Democracy

A Group Decision

- There is a group of people (>1)
- There are many options (>2)
- Everyone has different opinions on those options (a strict ordering of preferences)
- How do we choose?

Example: Movie Night

	Alice	Bob	Carol	Dave	Eve	Fred	Gina
Spirited Away	1	4	1	3	4	3	1
Howl's Moving Castle	2	2	2	2	3	2	3
My Neighbor Totoro	4	3	4	1	1	4	2
Ponyo	3	1	3	4	2	1	4

Aggregate
????
????
????
????

First Past The Post

	Alice	Bob	Carol	Dave	Eve	Fred	Gina
Spirited Away	**1**	4	**1**	3	4	3	**1**
Howl's Moving Castle	2	2	2	2	3	2	3
My Neighbor Totoro	4	3	4	**1**	**1**	4	2
Ponyo	3	**1**	3	4	2	**1**	4

Aggregate
1
????
????
????

First Past The Post

	Alice	Bob	Carol	Dave	Eve	Fred	Gina
Spirited Away	1	4	1	3	4	3	1
Howl's Moving Castle	**2**	2	**2**	2	3	2	3
My Neighbor Totoro	4	3	4	**1**	**1**	4	**2**
Ponyo	3	**1**	3	4	2	**1**	4

Aggregate
1
????
2
????

First Past The Post

	Alice	Bob	Carol	Dave	Eve	Fred	Gina
Spirited Away	1	4	1	3	4	3	1
Howl's Moving Castle	**2**	2	**2**	**2**	3	2	**3**
My Neighbor Totoro	4	3	4	1	1	4	2
Ponyo	3	**1**	3	4	**2**	**1**	4

Aggregate
1
3
2
????

What You Can't Do With Math #2: Democracy

First Past The Post

	Alice	Bob	Carol	Dave	Eve	Fred	Gina
Spirited Away	1	4	1	3	4	3	1
Howl's Moving Castle	2	2	2	2	3	2	3
My Neighbor Totoro	4	3	4	1	1	4	2
Ponyo	**3**	**1**	**3**	**4**	**2**	**1**	**4**

Aggregate
1
3
2
4

First Past The Post

	Alice	Bob	Carol	Dave	Eve	Fred	Gina
Spirited Away	1	4	1	3	4	3	1
Howl's Moving Castle	2	2	2	2	3	2	3
My Neighbor Totoro	4	3	4	1	1	4	2
Ponyo	3	1	3	4	2	1	4

Aggregate
1
3
2
4

Instant Runoff

	Alice	Bob	Carol	Dave	Eve	Fred	Gina
Spirited Away	**1**	4	**1**	3	4	3	**1**
Howl's Moving Castle	2	2	2	2	3	2	3
My Neighbor Totoro	4	3	4	**1**	**1**	4	2
Ponyo	3	**1**	3	4	2	**1**	4

Aggregate
????
4
????
????

Instant Runoff

	Alice	Bob	Carol	Dave	Eve	Fred	Gina
Spirited Away	1	4	1	3	4	3	1
Howl's Moving Castle	2	2	2	2	3	2	3
My Neighbor Totoro	4	3	4	1	1	4	2
Ponyo	3	1	3	4	2	1	4

Aggregate
????
4
3
????

Instant Runoff

	Alice	Bob	Carol	Dave	Eve	Fred	Gina
Spirited Away	**1**	4	**1**	**3**	4	3	**1**
Howl's Moving Castle	2	2	2	2	3	2	3
My Neighbor Totoro	4	3	4	1	1	4	2
Ponyo	3	**1**	3	4	**2**	**1**	4

Aggregate
????
4
3
2

Instant Runoff

	Alice	Bob	Carol	Dave	Eve	Fred	Gina
Spirited Away	**1**	**4**	**1**	**3**	**4**	**3**	**1**
Howl's Moving Castle	2	2	2	2	3	2	3
My Neighbor Totoro	4	3	4	1	1	4	2
Ponyo	3	1	3	4	2	1	4

Aggregate
1
4
3
2

Instant Runoff

	Alice	Bob	Carol	Dave	Eve	Fred	Gina
Spirited Away	1	4	1	3	4	3	1
Howl's Moving Castle	2	2	2	2	3	2	3
My Neighbor Totoro	4	3	4	1	1	4	2
Ponyo	3	1	3	4	2	1	4

Aggregate
1
4
3
2

Borda Count

	Alice	Bob	Carol	Dave	Eve	Fred	Gina		Aggregate
Spirited Away	**1**	**4**	**1**	**3**	**4**	**3**	**1**	= 17	**2**
Howl's Moving Castle	**2**	**2**	**2**	**2**	**3**	**2**	**3**	= 16	**1**
My Neighbor Totoro	**4**	**3**	**4**	**1**	**1**	**4**	**2**	= 19	**4**
Ponyo	**3**	**1**	**3**	**4**	**2**	**1**	**4**	= 18	**3**

Comparison

	FPTP	IRV	Borda
Spirited Away	1	1	2
Howl's Moving Castle	3	4	1
My Neighbor Totoro	2	3	4
Ponyo	4	2	3

Defining the problem

- We wish to find a function that takes in individual preferences and outputs a global ranking

- Many different such functions exist

- Want to find the best one

Some baseline conditions we definitely need

- The only thing that determines the relative rankings of A and B in the final rankings is the relative rankings of A and B in the individual rankings (**independence of irrelevant alternatives**)

- No single person controls the outcome in every situation (**non-dictatorship**)

- If every single person prefers option A to option B, the overall ranking should put A above B (**unanimity**)

Arrow’s Impossibility Theorem:

There is no possible voting system that always satisfies these three conditions!

independence of irrelevant alternatives

non-dictatorship

unanimity

“Nah, no way, you’re messing with me”

- This is a very reasonable response!

- However I wouldn’t have said it if I couldn’t prove it!

- >:)

A Proof of Arrow's Impossibility Theorem: Intro

- Imagine we have a set of N people, each of which has a list of preferences

- Let's look at some specific cases of what the situation can look like, and then try to generalize facts from these to arbitrary cases

- We'll assume independence of irrelevant alternatives (IIA) and unanimity, with the goal of showing they imply dictatorship

The general layout

n_1	n_2	n_3	...	...	...	...	...	n_N
A	...	A	...	...	...	...	...	...
...	C	...	...	...	...	...	...	...
...	B	...	...	...	...	...	...	...
B	...	B	...	...	...	...	...	...
...	A	...	...	...	...	...	...	...
C	...	...	...	...	...	...	...	...
...	...	C	...	...	...	...	...	...

Society
...
A
...
C
...
B
...

Polarization conservation lemma

If everybody ranks B either at the top or bottom, society has to rank B either at the top or bottom

n_1	n_2	n_3	...	...	...	...	...	n_N
B	...	...	...	B	B	B	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	B	B	B	...	...	...	B	B

Society
B
...
A
...
...
C
...

Polarization conservation lemma

Assume A>B>C for some A, C.

n_1	n_2	n_3	...	...	...	...	...	n_N
B	...	...	...	B	B	B	...	...
...	...	C	...	...	...	...	A	...
A	...	...	C	A	...	A	...	A
...	C	...	...	...	C	...	C	...
C	A	...	A	...	...	C	...	...
...	...	A	...	...	A	...	...	C
...	B	B	B	C	...	...	B	B

Society
...
A
...
B
...
C
...

Lemma proved

n_1	n_2	n_3	...	...	...	...	...	n_N
B	...	...	...	B	B	B	...	...
...	...	A	...	...	...	...	C	...
C	...	...	A	C	...	C	...	C
...	A	...	...	...	A	...	A	...
A	C	...	C	...	...	A	...	...
...	...	C	...	...	C	...	...	A
...	B	B	B	A	...	...	B	B

Assume A>B>C for some A, C. Switching A and C should keep A>B and B>C by IIA, but also switching the labels should keep C>B>A → contradiction

Society
...
C
...
B
...
A
...

Pivotal voter argument

n_1	n_2	n_3	...	...	...	...	...	n_N
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
B	B	B	B	B	B	B	B	B

Suppose now that everybody ranks B last. By unanimity, society must as well.

Society
...
...
...
...
...
...
B

Pivotal voter argument

n_1	n_2	n_3	...	...	...	...	...	n_N
B	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	B	B	B	B	B	B	B	B

Now, starting with voter 1, start flipping people from putting B at the bottom to putting B at the top

Society
...
...
...
...
...
...
B

Pivotal voter argument

n_1	n_2	n_3	...	...	...	...	...	n_N
B	B	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	B	B	B	B	B	B	B

Now, starting with voter 1, start flipping people from putting B at the bottom to putting B at the top

Society
...
...
...
...
...
...
B

Pivotal voter argument

n_1	n_2	n_3	...	...	...	...	...	n_N
B	B	B	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	B	B	B	B	B	B

Now, starting with voter 1, start flipping people from putting B at the bottom to putting B at the top

Society
...
...
...
...
...
...
B

Pivotal voter argument

n_1	n_2	n_3	...	...	...	...	...	n_N
B	B	B	B	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	B	B	B	B	B

Now, starting with voter 1, start flipping people from putting B at the bottom to putting B at the top

Society
...
...
...
...
...
...
B

Pivotal voter argument

n_1	n_2	n_3	...	n_*	...	...	...	n_N
B	B	B	B	B	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	B	B	B	B

For some voter in the list, making this switch will result in society switching the placement of B from last to first (by the lemma we just proved)

Society
B
...
...
...
...
...
...

Dictatorship of n_* over A, C

n_1	n_2	n_3	...	n_*	...	...	...	n_N
B	B	B	B	B	...	...	...	...
...	...	...	A	...	...	A	...	...
A	...	...	...	...	A	...	A	...
...	A	...	C	...	...	C	...	A
C	...	...	...	...	...	...	...	C
...	C	C	...	...	C	...	C	...
...	...	A	...	...	B	B	B	B

Consider A and C not equal to B. Assign arbitrary preferences to all individuals except n_*.

Society
B
...
...
...
...
...
...

Dictatorship of n_* over A, C

n_1	n_2	n_3	...	n_*	...	...	...	n_N
B	B	B	B	C	...	...	...	...
...	...	...	A	B	...	A	...	...
A	...	...	...	...	A	...	A	...
...	A	...	C	...	...	C	...	A
C	...	...	...	A	...	...	...	C
...	C	C	...	...	C	...	C	...
...	...	A	...	...	B	B	B	B

Now, imagine n_* puts C in the top spot above B.

Society
B
...
...
...
...
...
...

Dictatorship of n_* over A, C

n_1	n_2	n_3	...	n_*	...	...	...	n_N
B	B	B	B	C	...	...	...	...
...	...	...	A	B	...	A	...	...
A	...	...	...	...	A	...	A	...
...	A	...	C	...	...	C	...	A
C	...	...	...	A	...	...	...	C
...	C	C	...	...	C	...	C	...
...	...	A	...	...	B	B	B	B

By IIA, the relative rankings of C and B should be the same as if B were at the bottom, in which case society puts B at the bottom, so C>B

Society
C
B
...
...
...
...
...

Dictatorship of n_* over A, C

n_1	n_2	n_3	...	n_*	...	...	...	n_N
B	B	B	B	C	...	...	...	...
...	...	...	A	B	...	A	...	...
A	...	...	...	...	A	...	A	...
...	A	...	C	...	...	C	...	A
C	...	...	...	A	...	...	...	C
...	C	C	...	...	C	...	C	...
...	...	A	...	...	B	B	B	B

Similarly, IIA tells us the rankings of A and B are the same as if B were at the top, so B>A

Society
C
B
...
...
A
...
...

Dictatorship of n_* over A, C

n_1	n_2	n_3	...	n_*	...	...	...	n_N
...	...	...	...	...	...	...	...	...
...	...	...	A	...	...	A	...	...
A	...	...	...	C	A	...	A	...
...	A	...	C	...	...	C	...	A
C	...	...	...	A	...	...	...	C
...	C	C	...	...	C	...	C	...
...	...	A	...	...	...	...	...	...

So, C > A. By IIA, this should be true regardless of n_*'s ranking of B, so society's relative preference of A and C is always equal to n_*'s

Society
...
...
C
...
A
...
...

Dictatorship of n_* over A, C

n_1	n_2	n_3	...	n_*	...	...	...	n_N
...	...	...	...	...	...	...	...	...
...	...	...	A	...	...	A	...	...
A	...	...	...	A	A	...	A	...
...	A	...	C	...	...	C	...	A
C	...	...	...	C	...	...	...	C
...	C	C	...	...	C	...	C	...
...	...	A	...	...	...	...	...	...

Society
...
...
A
...
C
...
...

Dictatorship of n_{**} over X, Y

n_1	n_2	n_3	...	n_{**}	...	...	...	n_N
A	A	A	A	A	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	X	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	Y	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	A	A	A	A

Use a similar argument to show that there must be another element n_{**} who is a dictator over all pairs X, Y not including A

Society
A
...
X
...
Y
...
...

$n_{**} = n_{*}$

n_1	n_2	n_3	...	$n_{**} = n_{*}$	...	...	...	n_N
A	A	A	A	A	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	X	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	Y	...	...	...	...
...	...	...	...	...	...	...	...	...
...	...	...	...	...	A	A	A	A

n_{**} can move A from the top to the bottom, meaning n_{**} has power over the relative rankings of A and C for A, C not equals B, so n_{**} must be n_{*}

Society
A
...
X
...
Y
...
...

Contradiction found!

- Assuming IIA and unanimity implied a dictator → our 3 basic conditions were in fact mutually exclusive!

- It’s easy to get lost in the argument and think things only apply for those specific cases we generated -- in fact, this is a general statement. The only voting system that always satisfies IIA and unanimity is a dictatorship

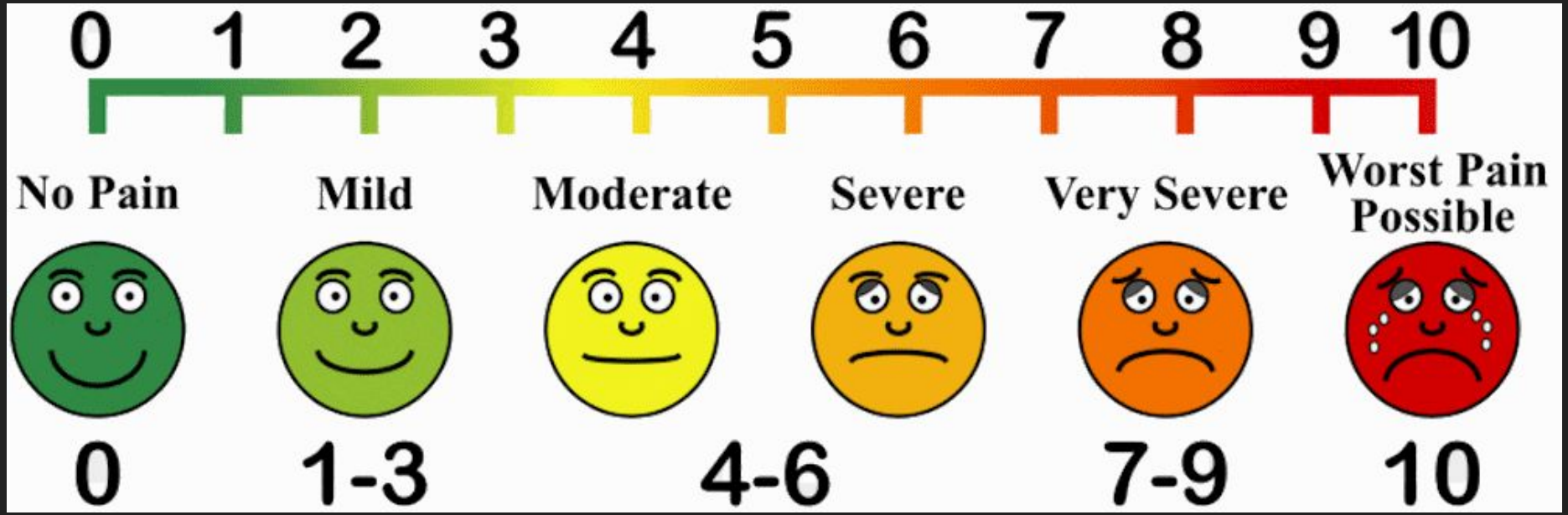
Further results

- It’s possible to show that, given random voter preferences, there is a lower bound for the chance of nontransitive outcomes

- Only applies where we care about all the output rankings; if we only care about first place Gibbard’s theorem shows that every voting system will incentivize dishonest voting

- (and in fact, even more general voting systems i.e. approval voting are shown to necessarily incentivize tactical voting)

How sad does this make you?



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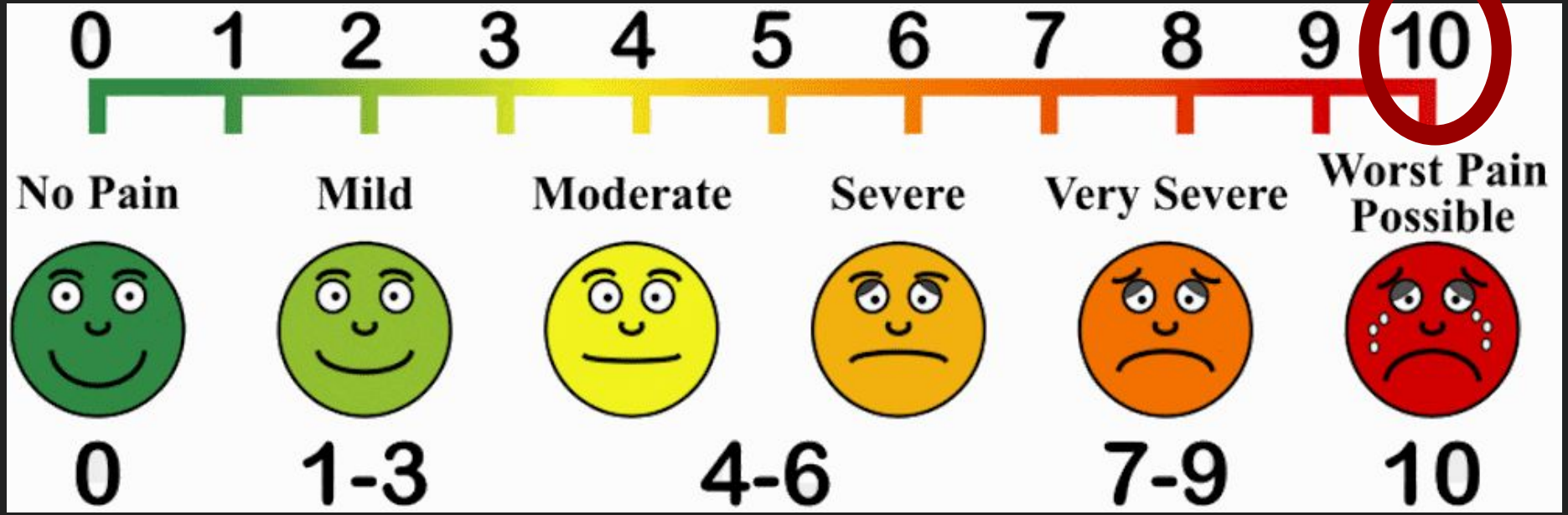

What You Can’t Do With Math #3: Math

We Must Know. We Will Know.

- It’s the early 1900s -- the world of math is exciting and promising

- Finding paradoxes from sloppy definitions (“does the set of all sets that don’t contain themselves contain itself?”) mathematicians like Russell, Cantor, Peano, etc begin laying careful axiomatic definitions of math

- David Hilbert’s program aimed to show that this set of axioms were consistent and could prove any true mathematical statement

Oh no it’s Austrian logician Kurt Gödel here to destroy everything you love and care about

The Claims

- First Incompleteness Theorem: every set of consistent (i.e. no contradictions) axioms is incomplete, meaning that there are true statements that are impossible to prove

- Second Incompleteness Theorem: it is impossible to use a set of axioms to prove their own consistency

- There's almost certainly not going to be time at this point, so I'll go through the proof quite hand-wavily :(

Gödel Numbering

- Consider all symbols defined in the formal system your axioms live in

- Associate every string of those symbols by a unique number (Gödel uses some wacky stuff with primes to huge powers for this)

Constant sign	Gödel number	Usual Meaning
~	1	not
∨	2	or
⊃	3	if...then...
∃	4	there is an...
=	5	equals
0	6	zero
s	7	the successor of
(	8	punctuation mark
)	9	punctuation mark
,	10	punctuation mark
+	11	plus
×	12	times

Gödel Numbering

- Now, any possible mathematical statement can be written as an integer (although some integers are meaningless)

- Additionally, for any statement, since our logical system has defined rules of deduction we can represent the proof or disproof of the statement as a sequence of numbers that follow these rules

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Gödel Numbering

- The question of whether a statement is provable is now an arithmetic one; it's a question about the existence of such a sequence of numbers following set arithmetic rules

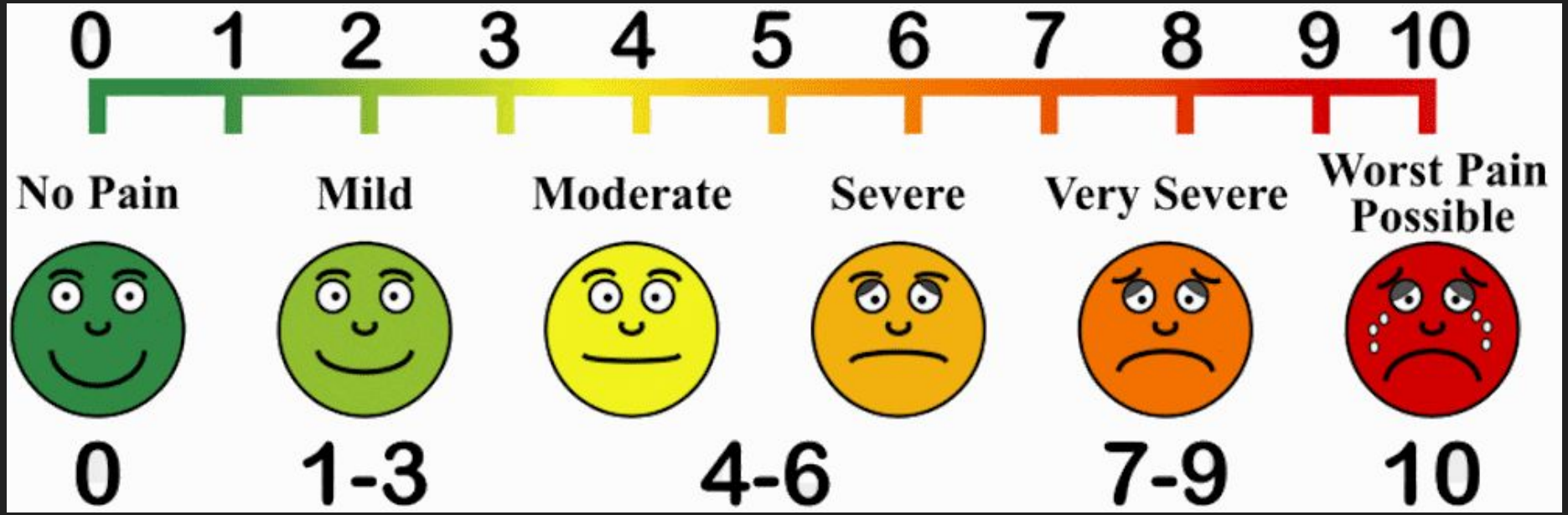
- So, we can encode with a Gödel number a statement meaning "the statement with Gödel number N is provable"

- But then, we can construct statements that reference their own provability. Gödel does this with a diagonalization argument.

Diagonal construction of unprovable statement

- Enumerate all formulas involving only one free variable as $B_1(n)$, $B_2(n)$, . . .
- Then, consider the formula "$B_n(n)$ is not provable"
- This is a formula involving only one free variable, so it belongs to our list at some index *k*
- But then, the statement $B_k(k)$ means "$B_k(k)$ is not provable", so we've built a statement that is true but unprovable

How sad does this make you?



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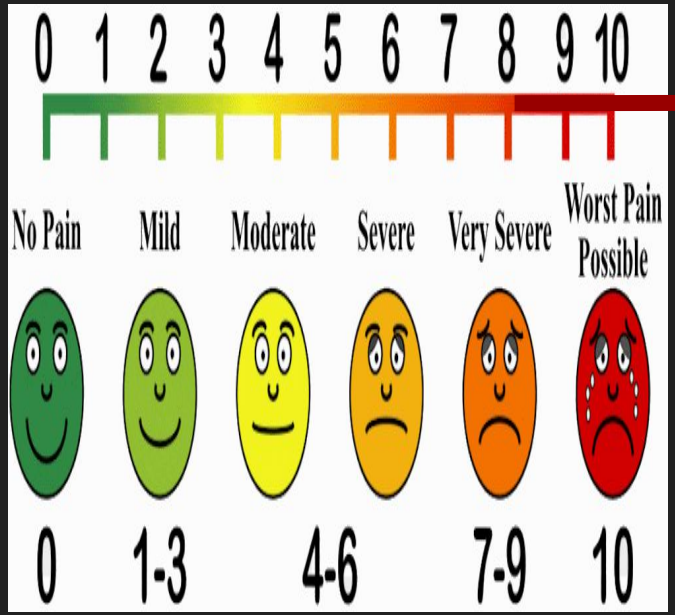
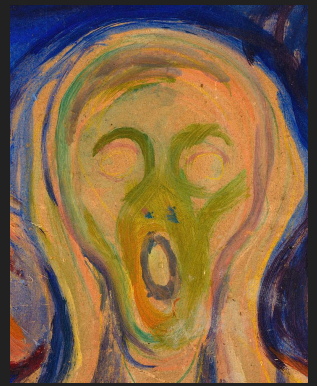


Godel # of unprovable tru statement

Witnessing the epistemological destruction of the very nature of existence; truth has no meaning and life has no purpose



The end!!!

I hope these results made you cry. If there’s time left, feel free to either vent about your feelings or ask questions now!