18.218 notes

Nathan Sheffield

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1 Logistics

- **Recordings:** Lectures will be recorded, but recordings are only available for people who have to miss the class for illnesses/whatever
- **Office Hours:** Tuesday after class 11:00-11:50.
- **Psets:** 5 psets; each with roughly 4 problems. We'll always have at least 2 weeks to do one. Collaboration is ok (as long as its bidirectional and writeups are independent). Start each problem solution with a list of the people you've discussed it with (even if this list is empty).
- **Grades:** Don't worry too much about grades. If you think there's a grading issue email yyao1@mit.edu.
- **Prereqs:** Nothing officially assumed, but you should probably know some math.
- I think all this stuff is in the syllabus tbh so maybe I didn't strictly need to write this down.

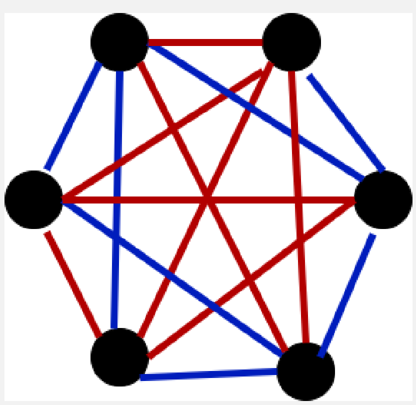
2 Introduction to Ramsey Theory

Ramsey theory really gets its name and its roots from **Ramsey's Theorem**:

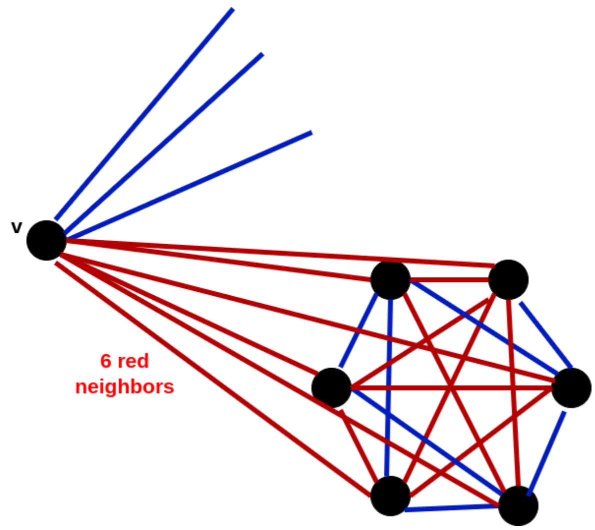
Ramsey's Theorem

Example: *Among any 6 people, there must either be 3 people who all know each other, or 3 people who all don't know each other*

For integers $k, l \geq 2$, there exists a positive integer $R(k, l)$ such that in any red-blue coloring of the edges of the complete graph on $R(k, l)$ vertices, there exists either a red k-clique or a blue l-clique.



Proof: We prove by induction on $k + l$. First, for $k = 2$ (argument applies symmetrically to $l = 2$), we can just let $R(2, l) = l$, since if no 2 vertices share a red edge, all edges must be blue. Now, suppose $k, l > 2$, and we already know the theorem for $k', l' \mid (k' + l') < (k + l)$. So, there exist some numbers $R(k - 1, l)$ and $R(k, l - 1)$. We claim that any graph of size $\mathbf{R(k-1,l)} + \mathbf{R(k,l-1)}$ must have a red k-clique or blue l-clique. To do so, imagine fixing a vertex in $K - (R(k - 1, l) + R(k, l - 1)$ By the pigeonhole principle, v must have at least $R(k - 1, l)$ red neighbours or $R(k, l - 1)$ blue neighbours. Assume wlog the first case. Now, either v's red neighbours have a l-blue clique (in which case we insta-win), or a $(k - 1)$-red clique, in which case we have a red k-clique with the inclusion of v.



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In this class, we will be interested in how large $R(k, l)$ has to be.

Definition: Ramsey Number

The Ramsey number $\mathbf{R}(\mathbf{k}, \mathbf{l})$ is the smallest number such that Ramsey's theorem holds.

- $R(3, 3) = 6$
- $R(4, 4) = 18$ (recently determined)
- $R(5, 5) =$ nobody knows!

Although we do not know an exact formula for $R(k, l)$, we can derive some bounds from our proof.

$$R(2, l) = l$$

$$R(k, l) \leq R(k - 1, l) + R(k, l - 1)$$

$$R(k, l) = R(l, k)$$

Solving this recurrence gives the following:

Erdős-Szekeres

For all $k, l \geq 2$,

$$R(k,l) \leq \binom{k+l-2}{k-1}$$

Note that this gives $R(k,k) \leq \binom{2k-2}{k-1} \leq 4^k$. Nobody knows a better-than-subexponential improvement on this bound (i.e. no 3.99^k bound is known). We have a $(\sqrt{2})^k$ lower-bound, but it's non-constructive. We have no superpolynomial lower-bound explicit construction.

3 Lower bounds for diagonal Ramsey numbers

The Ramsey numbers where both parameters are equal, $R(k,k)$, are known as **diagonal** Ramsey numbers. Last time it was mentioned a $(\sqrt{2})^k$ lower bound; now we'll show that bound. The proof uses the probabilistic method and is non-constructive. We have an easy constructive proof of $(k-1)^2$ by connecting $(k-1)$ blue cliques on $(k-1)$ vertices with red edges:

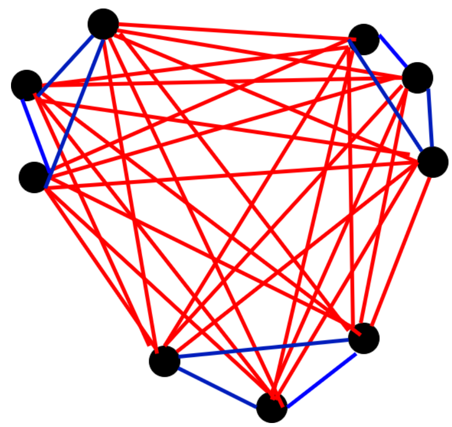


Figure 1: No monochromatic 4-clique exists.

Erdős 1947

For all $k \geq 2$,

$$R(k,k) \geq \sqrt{2}^k$$

Proof: First, note that for $k \leq 4$ the explicit bound from before is just as strong, so we have the claim for very small k.

Now, let $N = \sqrt{2}^k$, and consider a randomly-colored complete graph on N vertices (toss an independent fair coin to determine the color of each edge). For any given k vertices, there's a $2^{1-\binom{k}{2}}$ chance that they form a monochromatic clique. So, the expected number of monochromatic k-cliques is

$$\binom{N}{k} \cdot 2^{1-\binom{k}{2}} < \frac{N^k}{k!} 2^{1+\frac{k}{2}-\binom{k^2}{2}} \leq \frac{2^{k^2/2}}{k!} 2^{1+k/2-k^2/2} = \frac{2^{1+k/2}}{k!} \leq \frac{2^{1+k/2}}{2^{k-1}} = 2^{2-k/2}$$

For $k > 4$ this is less than 1, so there must be an outcome of the random coins with no monochromatic k-cliques. ■

In fact, by using Stirling's approximation instead of simply bounding $k!$ by 2^{k-1}, we can get $R(k,k) > \frac{k}{e\sqrt{2}}\sqrt{2}^k$. The best known lower bound is

$$R(k,k) > (1+o(1))\frac{2k}{\sqrt{2}e}\sqrt{2}^k$$

4 Hypergraphs

Recall that a **r-uniform hypergraph** is defined as a collection of size-r subsets ("edges") of some set V ("vertices"). The **complete** r-uniform hypergraph consists of all size-r subsets. A **clique** is a subset $S \subseteq V$ such that all possible r-subsets of S are edges.

Ramsey's Theorem for Hypegraphs

For any integers $r \geq 2$ and $k_1, \ldots, k_t \geq r$, there exists some integer $R_r(k_1, \ldots, k_t)$ such that, for any t-coloring of the r-uniform complete hypergraph on $R_r(k_1, \ldots, k_t)$ vertices, there is some i such that the graph has a k_i-clique of colour i.

The proof is very similar to the special case from before. We double-induct on both r and $(k_1 + \cdots + k_t)$ – that is to say, we induct on r on the outside, and $(k_1 + \cdots + k_t)$ on the inside.

We begin with an easy case. Suppose first that $k_i = r$ for some i. WLOG $k_t = r$. By induction we can assume that $R_r(k_1, \ldots k_{t-1})$ exists, so $R_r(k_1, \ldots, k_t)$ must also exist, because if there's no clique of size r in color t, there is no edge of color t.

Now, suppose that $k_1, \ldots, k_t > r$, and that we inductively assume the theorem for any $k'_1, \ldots, k'_t$ with $r' < r$, as well as for $(k'_1 + \cdots + k'_t) \leq (k_1 + \cdots + k_t)$ when $r' = r$. We claim that

$$|V| = R_{r-1}\Big(R_r(k_1 - 1, k_2, \ldots, k_{t-1}, k_t),\ R_r(k_1, k_2 - 1, \ldots, k_{t-1}, k_t),\ \ldots,\ R_r(k_1, \ldots, k_{t-1}, k_t - 1)\Big)$$

suffices for r, $k_1, \ldots, k_t$.

To show this claim, fix a vertex $v \in V$. For every $S \subseteq V$ with $|S| = r - 1$, consider the color assigned to $\{v\} \cup S$. This gives us a coloring of the complete $(r-1)$-uniform hypergraph on $V \setminus \{v\}$. By the induction hypothesis, for some i we can find a clique T of size $R_r(k_1, \ldots, k_i - 1, \ldots k_t)$ in this new colouring. Now, we apply the inductive hypothesis **again**. Looking at T back in the original colouring, we know that either it has a k_j-clique of colour j for some $j \neq i$ (in which case we win immediately), or it has a $k_i - 1$ clique of colour i, in which case we have a k_i clique with the inclusion of v. ■

Again, we can define the **hypergraph Ramsey number** $R_r(k, \ldots k_t)$ to be the smallest number for which the theorem is satisfied. The bound we get from our proof is absolute garbage; we'll see some marginally better (but still kinda garbage) bounds in a couple weeks.

5 Ramsey theory for point sets

Erdős-Szekeres Theorem on point sets in convex position

For every $k \geq 3$, there is some K such that:
Among any given K points in the plane such that no 3 points are colinear, you can find k points forming a convex k-gon.

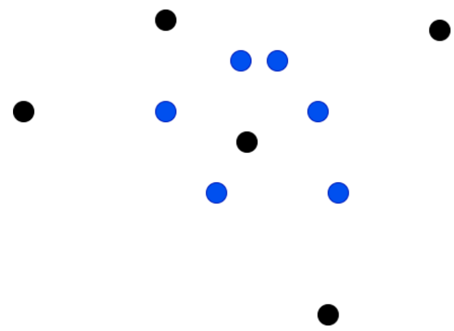


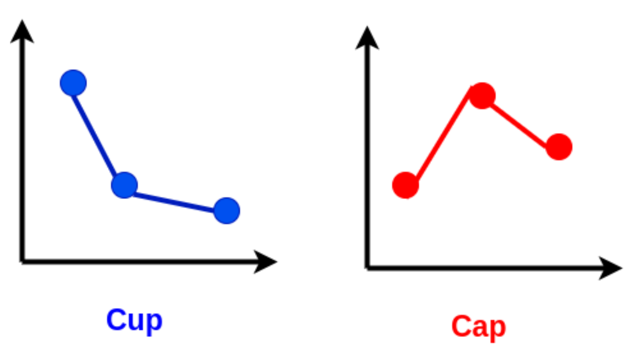
Figure 2: A set of non-colinear points in the plane containing a convex hexagon

[1]

Proof: We'll derive this result from Ramsey's theorem on hypegraphs. Looking at some set of points in the plane, we'll try to associate a hypergraph with it. Letting $r = 4$, we'll color a 4-tuple of points blue if they form a convex quadrilateral and red if they do not.

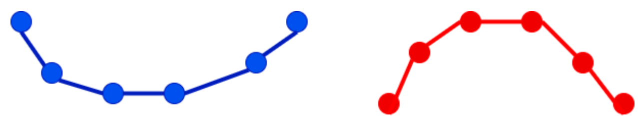
This gives a coloring of the complete 4-uniform hypergraph on those points. Now, imagine we set the number of points equal to $K = R_4(5, k)$. Among any 5 points there will be a size-4 subset that's a quadrilateral, so it will be impossible to find a red 5-clique. So, there must be a blue k-clique. This is a set of k points such that any four of them form a convex quadrilateral. If the convex hull of these points has an interior point, it lies on the interior of some triangle, so we have a contradiction to the claim that all sets of 4 points are convex. ∎

Alternative Proof: We assume without loss of generality that no line through the points is parallel to the y-axis (the property we care about is preserved by rotation). Triples of points have one of the following forms:



[1]Note that there's another Erdős-Szekeres theorem that's roughly equally famous which will come up on the homework

Fixing a set of points in the plane with the property that no line through them is parallel to the y-axis, we can define a colouring of the complete 3-hypergraph on them by colouring an edge red if its 3 points form a cap and blue if its three points form a cup. Then, if we started out with $K = R_3(k, k)$, we would be able to find either a set of k points such that any subset of 3 forms a cup, or such that any subset of 3 forms a cap. A set with this either of these properties is necessarily convex.



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6 Better bounds for convex point sets

Yet another Erdős-Szekeres result

For any $k, l \geq 3$, among any $\binom{k+l-4}{k-2} + 1$ non-colinear points in the plane (such that no line between them is parallel to the y axis), here are either k points forming a cap or l points forming a cap.

Proof: We proceed via induction without using Ramsey's theorem for graphs. If $k = 3$, there are $\binom{l-1}{1} + 1$ points. If no 3 points form a cap, all 3-tuples form cups, and so we can find a cup of size l. The theorem also holds for $l = 3$ by symmetry.

Now, let $k, l > 3$, and suppose we already have the result for $k' + l' < k + l$. Assume for contradiction that there is no cap of size k or cup of size l. Consider all points that are the rightmost point of some $k - 1$ cap. If we remove all of these points, there must no longer be a cap of size $k - 1$ (since we remove at least one point from every cap and we assume we didn't have a k cap to begin with). Since there's also no cup of size l by assumption, our inductive hypothesis tells us that there are at most $\binom{k+l-5}{k-3}$ points left, so we must have deleted at least $\binom{k+l-4}{k-2} + 1 - \binom{k+l-5}{k-3} = \binom{k+l-5}{k-2} + 1$ points. Looking at those deleted endpoints, we can now apply the inductive hypothesis again to find either a cap of size k or a cup of size $l - 1$. Finding a cap of size k is impossible by assumption, so there must be a cup of size $l - 1$ among these endpoints. But now, because the leftmost point in that cup is also the rightmost point of a $k - 1$ cup, we can either extend the $l - 1$ cup to an l cup or the $k - 1$ cap to a k cap. ■

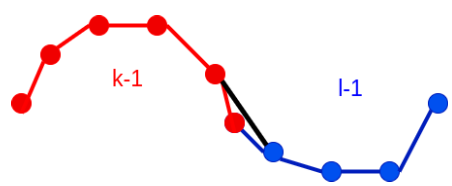

Figure 3: Depending on whether the additional vertex lies above or below the black line, we can either extend the cap or the cup by 1. In this case, we can extend the cup.

So, we have a $\binom{2k-4}{k-2} = O(4^k)$ upper bound on the number of points required for a convex k-gon to exist as a subset. The best lower bound known is $2^{k-2} + 1$, and the best known upper bound is $2^{k+o(k)}$ – trying to tighten this is known as the **Happy Ending Problem**.

7 Ramsey type results in arithmetic

Shur's Theorem (1916)

For every $t \geq 1$, there exists an N such that every t-colouring of $\{1, \ldots, N\}$, we can find x, y, z of the same colour satisfying $x + y = z$.

Proof: We use Ramsey's theorem for graphs. Let $N = R(3, 3, \ldots, 3)$, with t colours. If we colour each edge $(v, w) \in K_N$ with the color of $|v - w|$ in the original colouring of $\{1, \ldots, N\}$, then N is large enough that there must be a monochromatic triangle. Taking the original numbers corresponding to the edges in this graph, we find x, y, z of the same colour with $x + y = z$. ■

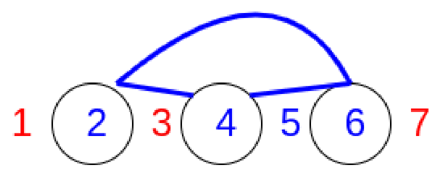

Figure 4: This triangle implies that $|4-2| = 2$, $|6-4| = 2$, $|2-6| = 4$ are all the same colour in the original colouring

Van der Waerden's Theorem (1927)

For every $t \geq 1$ and $k \geq 2$, there exists an N such that every t-colouring of $\{1, \ldots, N\}$, there exists a monochromatic arithmetic progression of length k.

This theorem has a nice generalization:

Hales-Jewett Theorem

Definition: A **combinatorial line** in $[k]^n$ corresponds to some $\lambda \in ([k] \cup \{\star\})^n$ with at least one $\star$. This corresponds to a subset of $[k]^n$ of size k obtained by replacing all $\star$'s with $1, \ldots, k$ respectively.

Theorem: For every $k, t \geq 1$ there exists some n such that every colouring of $[k]^n$ with t colours has a monochromatic combinatorial line.

Considering these n-tuples as k-ary representations of numbers, a combinatorial line of size k will correspond to an arithmetic progression of the following form:

```
1 0 0 3 7 0 2
1 1 0 3 7 1 2
1 2 0 3 7 2 2
1 3 0 3 7 3 2
1 4 0 3 7 4 2
1 5 0 3 7 5 2
1 6 0 3 7 6 2
1 7 0 3 7 7 2
1 8 0 3 7 8 2
1 9 0 3 7 9 2
```

So, letting $N = k^n$, the Hales-Jewett Theorem implies Van der Waerden's Theorem. Thus, Van der Waerden's theorem is given by the following proof:

Proof of Hales-Jewett Theorem: We proceed by induction on k. If $k = 1$ this is trivial (a combinatorial line is a single point). Now, fix $k \geq 2$, t, and assume the result for $k - 1$ and any number of colours. We now claim that for $1 \leq j \leq t$ there exists an n such that for every colouring of $[k]^n$ either

- There exists a monochromatic combinatorial line or
- There exist j combinatorial lines $L_1, \ldots, L_j$ with the same "endpoint[2]" x such that each of $L_1 \setminus x, L_2 \setminus x, \ldots, L_j$ are monochromatic and they have distinct colours.

when $j = t$, this implies that a monochromatic combinatorial line must exist, because we only have t distinct colours. We will prove the claim by induction starting from 1.

For $j = 1$, use the inductive hypothesis on k to take n such that the claim holds for $k - 1$ and t colours. Restricting our colouring of $[k]^n$ to $[k-1]^n$, we can find a monochromatic line for that restriction, which all the inner inductive hypothesis asks for when $j = 1$.

Now, fix $j \geq 2$ and assume we know the inner inductive hypothesis for $j - 1$. There's some n' satisfying the claim for $j - 1$; also let n'' be a value satisfying the Hales-Jewett theorem for $k - 1$ and $t^{k^{n'}}$ colors. Note that a coloring on $[k]^{n'+n''}$ with t colors gives a coloring of $[k]^{n''}$ with $t^{k^{n'}}$ colors, obtained by thinking of the colors for each of the first n' vertices in the original problem as encoding the name of a colour in the new problem.

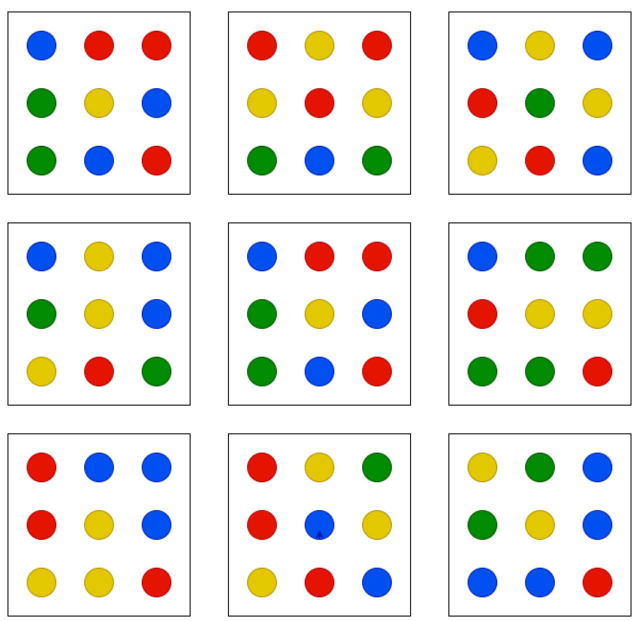

Figure 5: This is meant to depict a 4-colouring of $[3]^4$; the grid within a given box represents 2 of the dimensions, while the outer grid represents the other 2. Observe that, instead of thinking of of this as a 4-colouring of $[3]^4$, we could instead think of this as a 4^9-coloring of $[3]^2$ if we looked just at the outer grid and imagined the whole interior of the box corresponding to a colour. For example, the top left box corresponds to the colour "BRRGYBGBR".

By choice of n'', there is a combinatorial line L in $[k]^{n''}$ with endpoint y such that $L \setminus \{y\}$ is monochromatic. The color of $L \setminus \{y\}$ corresponds to a coloring $\phi : [k]^{n'} \to [t]$. By choice of n' we can find $j-1$ differently-coloured length $k - 1$ combinatorial lines with the same endpoint within this colouring.

[2] defined as the point where $\star$ is replaced with k

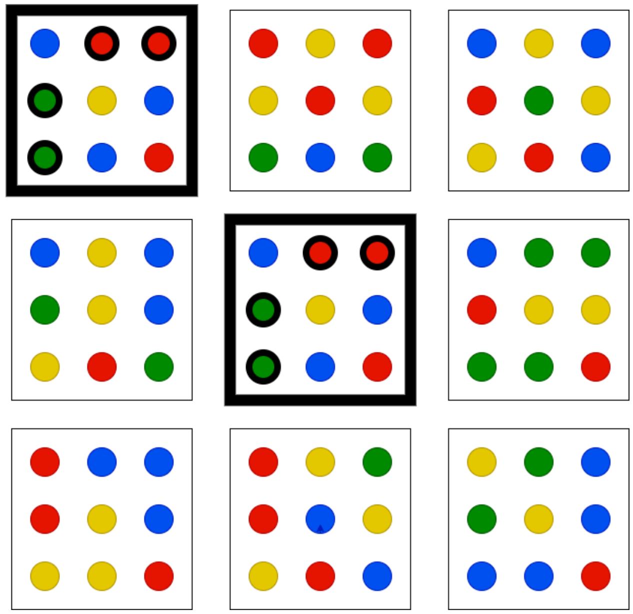

Figure 6: The black squares correspond to the length $k-1$ combinatorial line L in the outer colouring, and the black circles correspond to our $j-1$ (here imagine $j=3$) shared-endpoint combinatorial lines in the inner colouring.

Now, consider the colour of the shared endpoint of the $j-1$ lines within the inner colouring. That is, the blue circle in the top left of the bolded boxes in our example. If this shares a colour with any of the $j-1$ lines ending there in the inner colouring, we have a length k line and we are done. So, we can assume it's of a distinct colour (hence why it is blue here). But now, taking that points across all $k-1$ copies of this inner colour in L gives a length $k-1$ combinatorial line in the overall colouring. By looking at our $j-1$ combinatorial lines in the inner colouring and "diagonalizing" them across the overall colouring, this lets us find j distinctly-coloured combinatorial lines in the overall colouring with the same endpoint.

Van der Waerden's theorem (density version)

For every $k \geq 2$, $\epsilon > 0$, for all sufficiently large N, for any subset $A \subseteq N$, $|A| \geq \epsilon N$, A contains an arithmetic progression of length k.

Proof: Something with hypergraph regularity that we're not going to prove in class but that you, the reader, should look up on your own.

8 Off-diagonal Ramsey numbers

8.1 Upper bounds

Recall that

$$R(k,l) \leq \binom{k+l-2}{k-1}$$

For fixed l and large k, this is $O(k^{l-1})$. However, this is not tight – we know, for instance, that for $l=3$ $R(k,3) \in \Theta(\frac{k^2}{\log k})$. We present a result improving on this bound by a logarithmic factor (getting known tight bounds for $l=3$, but unknown for higher l).

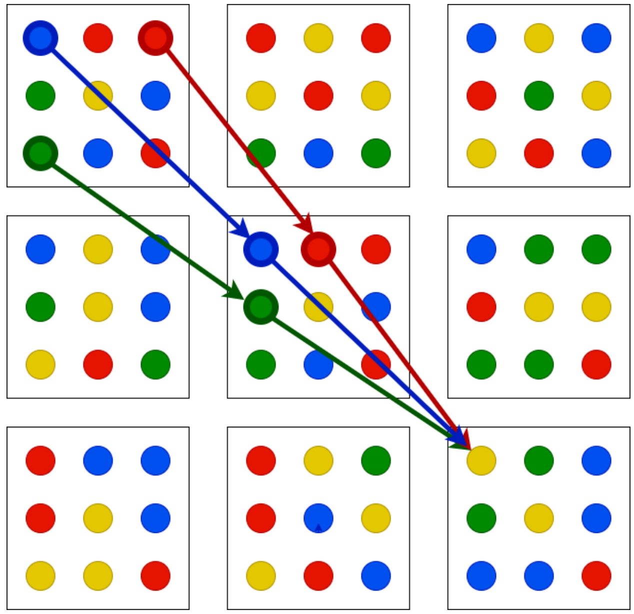

Figure 7: The $j-1$ lines in a single box can be diagonalized to $j-1$ lines in the overall colouring sharing an endpoint at the endpoint of the outer line L (depicted here by the red and green arrows). Then, reading along the endpoints in the inner colourings gives us a jth line (depicted here in blue).

Ajtai Komlos Szemeredi (1980)

For any $l \geq 3$,

$$R(k,l) \in O\left(\frac{k^{l-1}}{(\ln k)^{l-2}}\right)$$

We start by proving the $l = 3$ case, for which we present the following useful lemma:

Lemma: Every triangle-free graph G on n vertices with average degree d has independence number

$$\alpha(G) \geq n \cdot \frac{d \cdot \ln d - d + 1}{(d-1)^2}$$

Proof of lemma: It can be checked that $f(d) = \frac{d \cdot \ln d - d + 1}{(d-1)^2}$ is continuous, with negative first derivative and nonnegative second derivative. Note also that $(d+1)f(d) = 1 + (d - d^2)f'(d)$. We proceed by induction on n (note that $n = 0$ is trivial). We choose some vertex x, find the largest independent set in the graph $G_x = G \setminus (\{x\} \cup N(x))$ where $N(x)$ is the neighbourhood of x, and then add x to that set. For any vertex x, we can define

$$D(x) = \sum_{y \in N(x)} \deg(y) = \#\text{edges incident to at least one vertex in } \{x\} \cup N(x)$$

Now, we know that

$$|V(G_x)| = n - 1 - \deg(x)$$

$$|E(G_x)| = |E(G)| - D(x) = \frac{dn}{2} - D(x)$$

So G_x has average degree

$$d_x = \frac{2|E(G_x)|}{|V(G_x)|} = \frac{dn - 2D(x)}{n - 1 - \deg(x)}$$

By the induction hypothesis, G_x contains an independence set of size at least $(n-1-\deg(x)) \cdot f(d_x)$, so adding x gives that

$$\alpha(G) \geq \alpha(G_x) + 1 \geq (n - 1 - \deg(x)) \cdot f(d_x) + 1$$

Since this holds for any vertex x, it must also hold for the average x:

$$\alpha G \geq 1 + \sum_{x \in V(G)} \frac{n - 1 - \deg(x)}{n} \cdot f(d_x)$$

By convexity, $f(d_x)$ is at least $f(d) + (d_x - d) \cdot f'(d)$.

$$\alpha G \geq 1 + \sum_{x \in V(G)} \frac{n - 1 - \deg(x)}{n} \cdot \big(f(d) + (d_x - d) \cdot f'(d)\big)$$

$$= 1 + (n - 1 - d)\big(f(d) + (d_x - d) \cdot f'(d)\big) + \frac{1}{n} \sum_{x \in V(G)} (dn - 2D(x)) \cdot f'(d)$$

$$= 1 + (n - 1 - d)\big(f(d) + (d_x - d) \cdot f'(d)\big) + dnf'(d) - \frac{2f'(d)}{n} \sum_{x \in V(G)} D(x)$$

$$= 1 + (n - 1 - d)\big(f(d) + (d_x - d) \cdot f'(d)\big) + dnf'(d) - \frac{2f'(d)}{n} \sum_{x \in V(G)} \sum_{y \in N(x)} \deg(y)$$

$$= 1 + (n - 1 - d)\big(f(d) + (d_x - d) \cdot f'(d)\big) + dnf'(d) - \frac{2f'(d)}{n} \sum_{y \in V(G)} \deg(y)^2$$

$$\begin{aligned} &\geq 1 + (n - 1 - d)\big(f(d) + (d_x - d) \cdot f'(d)\big) + dnf'(d) - 2f'(d)d^2 \\ &= (n - 1 - d) \cdot f(d) + 1 + (d - d^2)f'(d) \\ &= (n - 1 - d) \cdot f(d) + (d + 1)f(d) \\ &= nf(d) \end{aligned}$$

■

Proof of $l = 3$ case from lemma: Suppose that for some $n > \frac{k^2}{\ln k - 1}$, there is a coloring of K_n with no red k-clique or blue triangle. Letting G be the graph of the blue edges, note that G is triangle free, and $\alpha(G) \leq k - 1$. We know the degree of any vertex is at most $k - 1$, since the neighbourhood of a vertex must form an independent set (if there was an edge between any two vertices in the neighbourhood we'd have a triangle). So, G has average degree $d \leq k$. We can also assume that the average degree $d \geq 10$ asymptotically (if not, we could find an independence set of size $\geq \frac{n}{40}$, which is not allowed for large n). Now, the lemma gives us

$$k - 1 \geq \alpha(G) \geq n \cdot \frac{d \cdot \ln d - d + 1}{(d-1)^2} \geq n \cdot \frac{\ln d - 1}{d} \geq n \cdot \frac{\ln k - 1}{k} \geq \frac{k^2}{\ln k - 1} \cdot \frac{\ln k - 1}{k} = k$$

which is contradiction. ■

Proof of general case: Let $C = 9999$. For $l = 3$, we've shown $R(k, 3) \leq \frac{k^2}{\ln k - 1} \leq \frac{C^2 k^2}{\ln k}$, so we will work from this as our base case, assuming $l \geq 4$ and the result for all $l' < l$. Suppose for contradiction that for

$n = \left\lfloor \frac{(Ck)^{l-1}}{(\ln k)^{l-2}} \right\rfloor$ there is a colouring of the edges of K_n without a red k-clique or blue l-clique. Let G be the graph of the blue edges. Then G does not have a clique of size l, and $\alpha(G) < k$. For every vertex $v \in V(G)$, the neighbourhood $N(v)$ must therefore have no clique of size $l-1$, and no independent set of size k. By the induction hypothesis, $|N(v)| \leq R(k, l-1) \leq \frac{(Ck)^{l-2}}{(\ln k)^{l-3}}$. For notational purposes, we will define $d = \frac{(Ck)^{l-2}}{(\ln k)^{l-3}}$ to be this degree upper-bound; note that this gives $|E(G)| \leq \frac{nd}{2}$.

Now, we go further. For any two adjacent vertices $v, w \in V(G)$, the common neighbourhood $N(v) \cap N(w)$ contains no clique of size $l-2$ (and no independent set of size k). So, by the induction hypothesis, $|N(v) \cap N(w)| \leq R(k, l-2) \leq \frac{(Ck)^{l-3}}{(\ln k)^{l-4}}$.

Note that $(\ln k)^4 < d$ and for $\varepsilon = \frac{1}{l-2}$, $(l-2)(1-\varepsilon) = l-3$, $(1-3)(1-\varepsilon) = l-4+\varepsilon$.

This gives that G contains at most $nd^{2-.99\varepsilon}$ triangles – to choose a triangle in G, we have n choices for the first vertex, at most d choices for the second, and at most $\frac{(Ck)^{l-3}}{(\ln k)^{l-4}} \leq d^{1-.99\varepsilon}$ choices for the third.
We now take a random subgraph of G in an attempt to find a triangle-free subgraph that lets us apply the lemma from before. We let G' include each vertex with independent probability $p = d^{\varepsilon/4-1}$.

$$\mathbb{E}[|V(G')|] = pn = nd^{\varepsilon/4-1}$$

$$\mathrm{Var}[|V(G')|] = p(1-p)n < pn$$

$$\text{By Chebyshev's inequality, } \Pr\Big[|V(G')| < \frac{1}{2}pn\Big] < \frac{1}{4}$$

$$\mathbb{E}[|E(G')|] = p^2|E(G)| = \frac{p^2nd}{2} \leq \frac{1}{2}nd^{\varepsilon/2-1}$$

$$\mathbb{E}[\#\text{triangles in } G'] = p^3(\#\text{triangles in } G) \leq p^3nd^{2-.99\varepsilon} \leq \frac{n}{d}$$

Using Markov's inequality and the union bound, the probabilities of the number of verticies, edges, and triangles all being within a factor of 4 of their expectations is at least $1/4$. Now, let $G'' \subseteq G'$ be the subgraph obtained by deleting one vertex from each triangle in G'. $|V(G'')| \geq \frac{1}{2}nd^{\varepsilon/4-1} - 4\frac{n}{d} \geq \frac{nd^{\varepsilon/4-1}}{4}$. G'' is triangle free with average degree $\frac{2|E(G'')|}{|V''|} \leq 16d^{\varepsilon/4}$, so our lemma from before gives $\alpha(G'') \geq |V''| \cdot \frac{\ln(16d^{\varepsilon/4})-1}{16d^{\varepsilon/4}} \geq \frac{1}{500} \cdot \frac{n}{d} \cdot \varepsilon \cdot \ln d$. More tweaking gets $\alpha(G'') \geq \frac{Ck}{1000} > k$, which gives a contradiction because we know $k > \alpha(G) \geq \alpha(G'')$.
■

The best known corresponding lower bound is that for every $l \geq 3$ there exists c_l s.t. $R(k,l) \geq c_l \left(\frac{k}{\ln k}\right)^{(l+1)/2}$ for all $k \geq 3$.

9 Bounds for Hypergraph Ramsey Numbers

The proof we gave about the existence of hypergraph Ramsey numbers gives an extremely weak upper bound. For instance, since $R_2(k,k)$ is exponential in k, our bound on $R_3(k,k)$ grows like $2 \uparrow\uparrow k$ which is like finite I guess but pretty gross. Fortunately, it is not the case that our bounds are just that bad – here's a slightly nicer statement:

Erdős Rado 1952

For any $k_1, \ldots, k \geq r \geq 3$ with $t \geq 2$ we have

$$R_r(k_1, \ldots, k_t) \leq t^{\binom{R_{r-1}(k_1-1, \ldots k_t - 1)}{r-1}}$$

Note that for $R_3(k,k)$ this gives a bound of $2^{2^{4k}}$, which is a lot better.

Proof of statement: Let $R = R_{r-1}(k_1 - 1, \ldots, k_t - 1)$, and consider a t-colouring of the complete r-uniform hypergraph on $t^{\binom{R}{r-1}}$ vertices. We want to show that for some i there exists a clique of size k_i monochromatic in i. To do so, we'll construct a sequence of vertices $v_1, \ldots v_R$ and subsets $S_1 \supseteq \cdots \supseteq S_R$ such that

- For any $1 \leq j_1 < \cdots \leq R$, the edges $\{v_{j_1}, \ldots, v_{j_{r-1}}\}$ have the same colour for all $v \in S_{j_{r-1}}$
- For every $j \in [R]$, we have $v_{j+1}, \ldots, v_R \in S_j$ but $v_1, \ldots, v_j \notin S_j$
- $|S_j| \geq t^{\binom{R}{r-1} - \binom{j}{r-1}} - j$ and $|S_R| \geq 1$

We construct this as follows:

- Let S_0 be the entire vertex set. Note that

$$|S_0| = t^{\binom{R}{r-1}} = t^{\binom{R}{r-1} - \binom{0}{r-1}} - 0$$

- For $j \in [R]$:
  - Pick an arbitrary index v_j in $S_j - 1$.
  - For any vertex $v \in S_{j-1} \cup \{v_j\}$, consider the colours of the edges $\{v_{j_1}, \ldots, v_{j_{r-2}}, v_j, v\}$ for each $(r-2)$-tuple $1 \leq j_1 < \ldots j_{r-2} \leq j - 1$. There are $\binom{j-1}{r-2}$ such tuples, and therefore $t^{\binom{j-1}{r-1}}$ subsets.
  - Choose the largest such subset to be S_j. You can show this gives the three conditions above; I don't really want to write it out.

■

There is a known lower bound that $k^{ck} < R_3(k,4)$; we will prove the slightly weaker statement that $2^{ck} < R_3(k,4)$.

The best known lower bound on $R_3(k,k)$ is 2^{ck^2} (on our pset). Showing whether the true behaviour is singly or doubly exponential could win you \$500! (Erdős offered a \$500 bounty.)

Erdős Hajnal 1972

There is an absolute constant c such that $R_3(k,4) \geq 2^{ck}$ for all $k \geq 3$.

Proof: Assuming k sufficiently large (we can adjust the constant to cover the smaller cases), we show that $c = \frac{1}{5}$ is sufficient.

Consider the complete 3-uniform hypergraph on $\lfloor 2^{k/5} \rfloor$ vertices coloured red and blue – we want to show that its possible theres no red k-clique or blue 4-clique. Choose a random tournament on the same vertex set (direction is independently random for each edge). Now, colour a hyperedge red if its vertices correspond to a cycle in the tournament, and blue if they correspond to a transitive tournament. There cannot be a blue clique of size 4, because you can't have four vertices in the tournament where any 3 form a cycle. We now bound the probability of having a red k-clique. For any k vertices, we have a red clique on those vertices if and only if they form a transitive tournament, which has probability $k! \cdot 2^{-\binom{k}{2}}$.

$$k! \cdot 2^{-\binom{k}{2}} \leq k^k 2^{-k^2+k/2} \leq 2^{-k^2/4}$$

There are $\binom{\lfloor 2^{k/5} \rfloor}{k}$ subsets of size k, so we can union bound and get total probability

$$\leq \binom{\lfloor 2^{k/5} \rfloor}{k} \cdot 2^{-k^2/4} \leq 2^{k^2/5} \cdot 2^{-k^2/4} = 2^{-k^2/20} < 1$$

■

Suppose now that we had more than 2 colours – we're trying to find $R_3(k, k, \ldots, k)$. The upper bound $2^{2^{c_t k}}$ extends for more colours, but when we have at least 4 colours we can prove a doubly exponential lower bound.

Erdős Hajnal

For all $k \geq 3$,

$$R_3(k, k, k, k) \geq 2^{R(k-1,k-1)-1} \geq 2^{2^{k/2-2}}$$

Proof: Let $R = R(k-1, k-1) - 1$, and consider a colouring of the edges of the complete graph G on $[R]$ without a monochromatic clique of size $k-1$. Now, consider the complete 3-uniform hypergraph on vertex set $\{0,1\}^R$.

Every vertex in this set corresponds to a binary number, which gives a natural ordering. Define $\delta(v, w)$ to be the maximum index of place value in which the numbers corresponding to v and w differ. Thus, if $v < w$, $v_{(\delta(v,w))} = 0$ and $w_{(\delta(v,w))} = 1$. So for $v < w < z \in \{0,1\}^R$, $\delta(v, w) \neq \delta(w, z)$, because this would require w to equal both 0 and 1 there. Also, note that for $v^{(1)} < v^{(2)} < \ldots v^{(l)}$, $\delta(v^{(1)}, v^{(l)}) = \max\left(\delta(v^{(1)}, v^{(2)}), \delta(v^{(2)}, v^{(3)}), \ldots, \delta(v^{(l-1)}, v^{(l)})\right)$.

Now, colour $v < w < z$:

- dark-red if $(\delta(v, w), \delta(w, t))$ is a red edge in G and $\delta(v, w) < \delta(w, z)$
- light-red if $(\delta(v, w), \delta(w, t))$ is a red edge in G and $\delta(v, w) > \delta(w, z)$
- dark-blue if $(\delta(v, w), \delta(w, t))$ is a blue edge in G and $\delta(v, w) < \delta(w, z)$
- light-blue if $(\delta(v, w), \delta(w, t))$ is a blue edge in G and $\delta(v, w) > \delta(w, z)$

We claim this colouring has no monochromatic clique of size k. Suppose there's a dark red clique of size k formed by $v^{(1)} < v^{(2)} < \cdots < v^{(k)}$. Since the clique is dark red, we have in particular that

$$\delta(v^{(1)}, v^{(2)}) < \delta(v^{(2)}, v^{(3)}) < \cdots < \delta(v^{(k-1)}, v^{(k)})$$

Since the edge $\{v^{(i)}, v^{(i+1)}, v^{(j+1)}\}$ is dark red, we have that $(\delta(v^{(i)}, v^{(i+1)}), \delta(v^{(j)}, v^{(j+1)}))$ is a red edge in G – but now we've found k vertices of G with a red $k-1$ clique, which contradictions our choice of R. The cases for the other colours are analogous.

■

We've been talking about 3-uniformity instead of general r because of the following:

Erdős Hajnal "Stepping Up Lemma"

For any $k_1, \ldots, k_t \geq r \geq 4$ with $t \geq 2$ we have

$$R_r(k_1, \ldots, k_t) \geq 2^{R_{r-1}\left(\left\lceil \frac{k_1-r+4}{2} \right\rceil, \left\lceil \frac{k_2-r+4}{2} \right\rceil, \ldots, \left\lceil \frac{k_t-r+4}{2} \right\rceil\right)-1}$$

This requires $r \geq 4$, so we can't get lower bounds for uniformity 3 from it, but it gives us nice bounds for $r > 3$ assuming we can find good ones for $r = 3$.

Proof: Let $R = R_{r-1}\left(\left\lceil \frac{k_1-r+4}{2} \right\rceil, \left\lceil \frac{k_2-r+4}{2} \right\rceil, \ldots, \left\lceil \frac{k_t-r+4}{2} \right\rceil\right) - 1$. Consider a colouring of the hyperedges of the complete $(r-1)$-uniform hypergraph H with vertex set $[R]$ and colours $[t]$ such that there is no clique of size $\left\lceil \frac{k_i-r+4}{2} \right\rceil$ for any $i \in [t]$. Now (as in the previous proof), we consider the complete r-uniform hypergraph on $\{0,1\}^R$, and aim to show a t-colouring without a k_i-clique for any i. Again, we define $\delta(v,w)$ to be the first index such that v and w differ. We define our hyperedge colouring as follows:

Consider a hyperedge $\{v^{(1)}, v^{(2)}, \ldots, v^{(r)}\}$; assume it's labeled such that $v^{(1)} < v^{(2)} < \cdots < v^{(r)}$. Let δ_i $\delta(v^{(i)}, v^{(i+1)})$ for $i = 1, \ldots, r-1$, and note that $d_i \neq d_{i+1}$ by our previous arguments.

- If our δ_i are monotonic – that is, $\delta_1 < \delta_2 < \cdots < \delta_{r-1}$ or $\delta_1 > \delta_2 > \cdots > \delta_{r-1}$, colour $\{v^{(1)}, v^{(2)}, \ldots, v^{(r)}\}$ with the colour of $\{\delta_1, \ldots, \delta_{r-1}\}$ in H.
- If $\delta_1 < \delta_2$ and $\delta_2 > \delta_3$, then colour $\{v^{(1)}, v^{(2)}, \ldots, v^{(r)}\}$ in colour 1
- If $\delta_1 > \delta_2$ and $\delta_2 < \delta_3$, then colour $\{v^{(1)}, v^{(2)}, \ldots, v^{(r)}\}$ in colour 2
- Otherwise, we can pick the colour arbitrarily (I will say colour 1 because that makes me happiest)

We will show there cannot be a clique of size k in colour 1 (this gives the statement for all the other colours by a symmetry argument). Suppose there exists such a clique $\{w^{(1)}, w^{(2)}, \ldots, w^{(k_1)}\}$ and let $\delta'_i = \delta(w^{(i)}, w^{(i+1)})$, and note that $\delta'_i \neq \delta'_{i+1}$.

We claim that, for some j, j' with $1 \leq j < j' \leq k_1 - r + 3$ and $j' - j \geq \left\lceil \frac{k_1-r+4}{2} \right\rceil - 1$ such that either $\delta'_j < \cdots < \delta_j$ or $\delta'_j > \cdots > \delta_j$. To show this claim, we note that there is no $i \in \{1, \ldots, k_1 - 1\}$ such that $\delta'_i > \delta'_{i+1}$ and $\delta'_{i+1} < \delta'_{i+2}$, because otherwise the edge $\{w^{(i)}, w^{(i+1)}, \ldots, w^{(i+r-1)}\}$ would have to have colour 2. So, the sequence $\delta'_1, \ldots, \delta'_{k-r+3}$ can be split into a monotonically increasing part followed by a monotonically decreasing part. Either the increasing or decreasing part will have at least $\left\lceil \frac{k_1-r+2}{2} \right\rceil$ steps.

But now, taking $\delta'_j, \delta'_{j+1}, \ldots, \delta'_{j'}$ in H, we note that they cannot form a clique of colour 1. So, there is some collection of $r-1$ indices $j \leq i_1 \leq i_2 \leq \cdots \leq i_{r-1} \leq j'$ such that $\{\delta'_{i_1}, \delta'_{i_2}, \ldots, \delta'_{i_{r-1}}\}$ does not have colour 1 in H. Look at the edge $\{w^{(i_1}, w^{(i_2)}, \ldots, w^{(i_{r-1}+1)}\}$. We have $\delta(w^{(i_m)}, w^{(i_m+1)}) = \delta'_{i_m}$ for all m here, since the δs are monotoniclly decreasing. So, this edge has the same colour as $\{\delta'_{i_1}, \delta'_{i_2}, \ldots, \delta'_{i_{r-1}}\}$, which is not colour 1, which gives a contradiction to the claim that we had a 1-clique. The increasing case is analogous. ■

10 Ramsey Numbers of graphs

For graphs $G_1, \ldots, G_t$, the Ramsey number $R(G_1, \ldots, G_t)$ is the smallest number such that for any t-colouring of the complete graph on $R(G_1, \ldots, G_t)$ vertices, there must exist a subgraph isomorphic to G_i in colour i for some i.

Note that, because any graph is a subgraph of the complete graph on that many vertices, $R(G_1, \ldots, G_t) \leq R(|V(G_1)|, \ldots, R(|V(G_t)|)$.

Chvatal Harary 1972

For any two connected graphs G and H,

$$R(G, H) \geq (\chi(G) - 1) \cdot (|V(H)| - 1) + 1$$

where $\chi(G)$ is the chromatic number of G.

The proof is a simple explicit construction left as an exercise.

Chvatal 1977

If T is a tree with $t \geq 1$ vertices and $s \geq 1$, then

$$R(K_s, T) = (s-1)(t-1) + 1$$

so we have exact bounds in this case.

Proof: The above lemma gives us the lower bound. Now, consider any red-blue colouring of the edges of a complete graph on $(s-1)(t-1)+1$ vertices. Suppose there's no blue s-clique. Let H be the subgraph of red edges; this has independence number at most s. Note that $\chi(H) \geq \frac{|V(H)|}{\alpha(H)} \geq \frac{(s-1)(t-1)+1}{(s-1)} > t-1$, but $\chi(H)$ is an integer so $\chi(H) \geq t$.

Now, we let H' be a **minimal** subgraph of H such that $\chi(H') \geq t$. We claim that every vertex in H' has degree $\geq t-1$. To see this, suppose there exists a vertex v with degree $\leq t-2$ – by minimality of H', we know $\chi(H' \setminus \{v\}) \leq t-1$, but a $(t-1)$-colouring of $H' \setminus \{v\}$ can be extended to a $(t-1)$-colouring of H' because v has at most $t-2$ neighbours.

But now, we can get any tree we want of size t as a subgraph of H, simply by greedily choosing vertices. Once we've assigned j vertices to our tree and we need to assign another child to vertex v, there will be at least $t-1$ edges out of v, and at most $j-1$ of them can be already used, so we have $t-j$ choices for what to add.

■

Chvatal Rodl Szemeredi Trotter

For any graph H with k vertices of maximum degree d,

$$R(H, H) \leq C_d \cdot k$$

, where C_d is a constant depending only on d.

Proof: We'll proceed by a couple of lemmas.

Lemma 1: Let H be a graph with k vertices and maximum degree d, and G a graph with at least $n = 4k$ vertices and density at least $1 - \frac{1}{8d}$. Then, H is a subgraph of G.
Proof: Let $\bar{G}$ be the complement of G. $\bar{G}$ has density at most $\frac{1}{8d}$, so has average degree at most $\frac{n}{8d}$. Thus $\bar{G}$ has at most $\frac{n}{2}$ vertices of degree more than $\frac{n}{4d}$. Let G' be the graph obtained by deleting all

such vertices from G. Now, we can find H as a subgraph of G' by greedily choosing vertices, noting that at any step there are only $\frac{n}{4}$ vertices with missing edges in G', and only at most $k \leq \frac{n}{4}$ that we've already chosen. So, there are always some remaining.

Lemma 2: We say that a graph is bi-(ρ, δ)-dense for $0 \leq \rho \leq 1$ and $0 \leq \delta \leq 1$ if for all pairs of disjoint subsets $X, Y \subseteq V(G)$ with $|X|, |Y| \geq \rho|V(G)|$, $d(X,Y) \geq \delta$. For a graph H with k vertices and maximum degree d: if G is bi-$\left(\frac{\delta^d}{8d^2}, \delta\right)$-dense on at least $8\delta^{-d}k$ vertices, then H is a subgraph of G.

Proof: First, observe that $\chi(H) \leq d+1$, so we can give H a $d+1$-colouring. Partition $V(G)$ into $U_1 \cup \cdots \cup U_{d+1}$, with $|U_i| \geq \left\lfloor \frac{|V(G)|}{d+1} \right\rfloor \geq \frac{|V(g)}{4d} \geq \delta^d k$. Now, we do a construction somewhat similar to the greedy one from last time. Let $v_1, \ldots, v_k$ be the vertices of H. We try to find a copy of H in G such that $v_1, \ldots, v_k \mapsto w_1, \ldots, w_k$, $w_i \in U_h$ for every vertex v_i of colour h. Suppose we've already embedded the first l vertices of H, and suppose the $l+1$th vertex of H has edges to N of those. Then, there are at least $\delta^N |U_h|$ vertices in U_h that are adjacent to all N of them.

Let $W_l, \ldots, W_k$ be the "candidate sets" for $w_l \ldots, w_k$ in the last condition. I fell off this train at some point.

Now, we prove that either Lemma 1 or Lemma 2 must apply.

Lemma 3: For any $0 \leq \rho \leq 1$ and $0 \leq \delta \leq \frac{1}{3}$ (letting s be an integer $\geq \frac{1}{\delta}$), for any graph G we have either

1. G has a bi-(ρ, δ)-dense subgraph on at least $\left(\frac{\rho}{2}\right)^s \cdot |V(G)|$ vertices
2. There is a vertex subset $U \subseteq V(G)$ of size $|U| \geq \left(\frac{\rho}{2}\right)^s \cdot |V(G)|$ with density $d(U) \leq 3\delta$

Proof: Let $t = \left\lceil \left(\frac{\rho}{2}\right)^s \cdot |V(G)| \right\rceil$ .Suppose G does not have a bi-(ρ, δ)-dense subgraph on at least t vertices. We will construct disjoint subsets $U_1 \ldots U_s \subseteq V(G)$ each of size t such that their union satisfies (2). We will do this by ensuring that every vertex v in a later U_i (or W, which is some subset of unassigned vertices of size at least $\left(\frac{\rho}{2}\right)^l \cdot |V(G)|$) has at most $2\delta t$ edges to any given earlier U_j. Proceed by induction, assuming we've found l such U_is (it is trivial for $l = 0$ so we have a base case). Denoting the unassigned vertex set W', we want to find $W, U_{l+1} \subseteq W'$ such that $|U_{l+1}| = t$, $|W| \geq \left(\frac{\rho}{2}\right) \cdot |W'|$ and every vertex in W has at most $w\delta t$ edges to U_{l+1}. Since we're not bi-(ρ, δ) dense, we can find disjoint subsets X and Y of W', each of size $\rho|W'|$, such that the density between them is low. Passing to subsets lets us reduce X to be of size t, and deleting the vertices in Y with more than twice average degree to X lets us ensure that all of them have low degree to X.

Now, we claim that the union of all these U_i has low density. We have density at most

$$\frac{s \cdot \binom{t}{2} + \binom{s}{2} \cdot 2\delta t^2}{\binom{st}{2}} = \frac{st(t-1) + s(s-1) \cdot 2\delta t^2}{st(st-1)} \leq 3\delta$$

The theorem now follows by letting $\delta = \frac{1}{24d}$, $\rho = \frac{\delta^d}{8d^2}$, and $s = \frac{1}{\delta} = 24d$. Let $C_d = \left(\frac{2}{\rho}\right)^{25d}$, and considering a red-blue edge colouring of the complete graph on $C_d \cdot k$ vertices. This bound is also known more generally for d-degenerate graphs, i.e. graphs that can be drawn one vertex at a time where you add at most d edges for each vertex.

■

Exciting new news! Campos Griffiths Mauns Sahasrabudhe

$$R(k,k) \leq (4-\epsilon)^k$$

for a constant ϵ (roughly equal to $\frac{1}{128}$).

11 Multi-colour Ramsey Numbers

We've proved that $R(k,\dots,k_t)$ exists – the upper bound from that proof is of the form

$$R(k,\dots,k_t) \leq \sum_{j=1}^{t} R(k_1,\dots,k_{j-1},k_j-1,k_{j+1},\dots,k_t$$

which gives

$$R(k_1,\dots,k_t) \leq \binom{(k_1+\cdots+k_t)-t}{k_1-1,k_2-1,\dots,k_t-1}$$

and in particular

$$R(k,\dots(t \text{ times})\dots,k) \leq t^{kt}$$

Fixing $k=3$, we have the following conjecture of Erdos:

Erdos will give you \$100 if you prove this! (not cliquebait)

There exists an absolute constant $c>0$ such that

$$R(3,\dots(t \text{ times})\dots,3) \leq C^t$$

The best known bounds are of the form

$$C_1^t \leq R(3,\dots(t \text{ times})\dots,3) \leq C_2 \cdot t!$$

Fixing some constant t, we can find the lower bound

$$R(k,\dots(t \text{ times})\dots,k) \leq t^{k/2}$$

by the same random colouring argument we did for $t=2$. For $t \geq 4$, we can get a better bound with a product construction.

Lemma:

$$R(k,\dots(t_1+t_2 \text{ times})\dots,k)-1 \geq (R(k,\dots(t_1 \text{ times})\dots,k)-1)\cdot(R(k,\dots(t_2 \text{ times})\dots,k)-1)$$

Proof: Make groups of $(R(k,\dots(t_2 \text{ times})\dots,k)-1)$, colouring within a group using a bad colouring of t_2, and between the groups using a bad colouring for t_1. ■

Together with $R(k,k) \geq 2^{k/2}$, this gives

$$R(k,\dots(t \text{ times})\dots,k) \geq 2^{kt/4}$$

for even k. Using the $R(k,k,k) \geq 3^{k/2}$ lower bound, we get

$$R(k,\dots(t \text{ times})\dots,k) \geq 3^{kt/6}$$

for t divisible by 3.

There was a recent improvement on this

Conlon-Ferber and Wigderson 2020

For any fixed $t \geq 2$, we have

$$R(k, \ldots (t \text{ times}) \ldots, k) \geq 2^{\left(\frac{3}{8} - \frac{1}{4} - o(1)\right) \cdot k}$$

Will Sawin improved this in 2021 to

$$R(k, \ldots (t \text{ times}) \ldots, k) \geq 2^{\left(0.393796(t-2) + \frac{1}{2} - o(1)\right) \cdot k}$$

We will prove the $\frac{3}{8}$ version with Will Sawin's approach.

Proof: Define $q_G(k)$ for a graph G and $k \geq 2$ as the probability that, choosing k vertices $v_1, \ldots, v_k$ independently and uniformly at random (with replacement), $v_1, \ldots, v_k$ form an independent set in G.

Lemma 1: For every $k, t \geq 2$, if a graph G contains no clique of size k,

$$R(k, \ldots (t \text{ times}) \ldots, k) \geq \left(\frac{1}{q_G(k)}\right)^{(t-2)/k} \cdot 2^{(k-1)/2}$$

Proof: Let $n = \left\lfloor \left(\frac{1}{q_G(k)}\right)^{(t-2)/k} \cdot 2^{(k-1)/2} \right\rfloor$. We want to find a t-colouring of K_n with no monochromatic k-clique. Consider $t-2$ independent uniformly random functions

$$f_1, \ldots, f_{t-2} : [n] \to V(G)$$

Colour an edge in K_G in colour i if f_i maps its two endpoints to vertices that have an edge in G. For the remaining edges, choose the colour at random to be either $t-1$ or t. Note that there cannot be a monochromatic clique in one of the first $t-2$ colours, because that would let us find a clique in G. Now, we consider the probability that a given k-vertex subset forms a monochromatic clique of colour $t-1$ or t. The probability that none of the edges get assigned one of the first $t-2$ colours is equal to the probability that each f_i maps the vertices to an independent set, which is precisely $\left(q_G(k)\right)^{t-2}$, and then the probability given this that it's either all $t-1$ or all t is $2^{1-k(k-1)/2}$, so the expected number of cliques is

$$\leq \binom{n}{k} \cdot \left(q_G(k)\right)^{t-2} \cdot 2^{1-k(k-1)/2} < n^k \cdot \left(q_G(k)\right)^{t-2} \cdot 2^{1-k(k-1)/2} \leq 1$$

■

Lemma 2: For every $k \geq 2$, there's a graph G not containing a clique of size k such that

$$q_G(k) \leq 2^{-\left(\frac{3}{8} - o(1)\right) k^2}$$

Proof: Let $m = \left\lceil 2^{(k-1)/2} \right\rceil$, and consider a random graph G on m vertices with each edge present or not present with independent probability $\frac{1}{2}$. We would like to show

$$\mathbb{E}\left[q_G(k) \mid G \text{ has no clique of size } k\right] \leq 2^{-\left(\frac{3}{8} - o(1)\right) \cdot k^2}$$

Since then averaging will give us our result. Note that

$$\mathbb{E}\left[q_G(k) \mid G \text{ has no clique of size } k\right] \leq \frac{\mathbb{E}\left[q_G(k)\right]}{\Pr\left[G \text{ has no clique of size } k\right]}$$

Now, union bounding gives us

$$\Pr\left[G \text{ has a clique of size } k\right] \leq \binom{m}{k} \cdot 2^{-\binom{k}{2}} \leq \frac{m^k}{k!} \cdot 2^{-k(k-1)/2} \leq \frac{\left(2^{(k-1)/2}+1\right)^k}{k!} \cdot 2^{-k(k-1)/2} \leq \frac{1}{2}$$

for sufficiently large k. This factor of 2 can get absorbed into our $o(1)$, so now we just need to show

$$\mathbb{E}\left[q_G(k)\right] \leq 2^{-\left(\frac{3}{8}-o(1)\right)\cdot k^2}$$

Letting $v_1, \ldots, v_k$ be independent uniformly random vertices of G, then

$$\mathbb{E}\left[q_G(k)\right] = \Pr\left[\{v_1, \ldots, v_k\} \text{ is an independent set}\right]$$

where the probability is over both the vertices and the edges. This is equal to

$$\sum_{l=1}^{k} \Pr\left[|\{v_1, \ldots, v_k\}| = l\right] \cdot 2^{-\binom{l}{2}}$$

$$\leq \sum_{l=1}^{k} \binom{m}{l} \cdot \left(\frac{l}{m}\right)^k \cdot 2^{-\binom{l}{2}}$$

$$\leq k^k \sum_{l=1}^{k} m^{l-k} \cdot 2^{-l(l-1)/2}$$

$$\leq k^k \sum_{l=1}^{l} 2(k-1)(l-k)/2 \cdot 2^{-l(l-1)/2}$$

$$\leq k^k \cdot 2^{-k(k-1)/2} \sum_{l=1}^{k} 2^{l(k-l)/2}$$

$$\leq k^k \cdot 2^{-\frac{1}{2}k^2+\frac{1}{2}k} \cdot k \cdot k^{\frac{k^2}{8}}$$

$$\leq 2^{-\left(\frac{3}{8}-o(1)\right)\cdot k^2}$$

■

Plugging the graph from lemma 2 into lemma 1, we get

$$R(k, \ldots t \text{ times} \ldots, k) \geq 2^{\left(\frac{3}{8}-o(1)\right)\cdot(t-2)\cdot k} \cdot 2^{(k-1)/2} = 2^{\left(\frac{3}{8}t-\frac{1}{4}-o(1)\right)\cdot k}$$

■

Induced Ramsey Theorem for graphs

For every graph H, there exists a graph G such that, for any red-blue colouring of the edges of G, there is a monochromatic **induced** subgraph of G isomorphic to H.

Proof: As with the original Ramsey theorem, we'll extend the statement to the off-diagonal case for the induction (given two graphs H_r and H_b, there exists a graph G such that any colouring of G contains either a red induced subgraph isomorphic to H_r, or a blue induced subgraph isomorphic to H_b). We proceed by induction on $|V(H_r)| + |V(H_b)|$. Note that this is trivial if H_r or H_b has an isolated vertex.

Now, fix a vertex v_r in H_r, and let $H_r' = H_r \setminus \{v_r\}$. Similarly, fix v_b in H_b and let $H_b' = H_b \setminus \{v_r\}$. We'll use the induction hypothesis to find graphs $G_1 = G(H_r, H_b')$ and $G(H_r', H_b)$ satisfying the theorem. Let $W_1, \dots, W_n$ be an enumeration of all vertex subsets of G_1 corresponding to induced subgraphs isomorphic to H_b'. Fix an isomorphism $H_b \to G_1[W_j]$ for each W_j, and let $W_j' \subseteq W_j$ be the image of the neighbourhood of v_b under that isomorphism.

Now, we prove the following claim by induction (completing the proof when $j = n + 1$):

Claim: For any $j \in [n+1]$, there exists a graph G_j and disjoint non-empty vertex subsets $U(v) \subseteq V(G_j)$ for all $v \in V(G_1)$ satisfying

- For distinct vertices v, v' in $V(G_1)$, we have $(u, u') \in E(G_j) \iff (v, v') \in E(G_1)$ for all $u \in U(v)$ and $u' \in U(v')$.
- For any red-blue colouring of G_j, we have one of the following:
  1. There is a red induced subgraph of G_j isomorphic to H_r
  2. There is a blue induced subgraph of G_j isomorphic to H_b
  3. For some $k \in \{j, \dots, n\}$ there is a blue induced subset of G_j consisting of exactly one vertex from each $U(v)$ for $v \in W_k$.

Proof: In the case $j = 1$, we can take G_1 and let $U(v) = \{v\}$ for all $v \in V(G_1)$, and note that this satisfies the claim. Now, suppose we know the claim for some $j \in [n]$. We construct G_{j+1} from G_j. First, let $U = \bigcup_{v \in W_j'} U(v) \subseteq V(G_j)$ and replace every vertex of G_j in U by a copy of $G(H_r', H_b)$ to define an new graph G_j^*. For any $u \in U$, let Φ_u be the collection of all embeddings $\phi : H_r' \to G_j^*$ in a copy of $G(H_r', H_b)$ corresponding to U. For every choice of a tuple of maps $(\phi_u)u \in U$, $\phi_u \in \Phi_u$, we add an additional vertex to G_j^* with edges to all vertices in $\bigcup_{u \in U} \phi_u N(v_r))$. Call this new graph G_{j+1} – we can verify that the conditions in the inductive hypothesis hold.

12 Ramsey Graphs

Note that our earlier results imply that every graph on n vertices has either a clique or independent set of size $\frac{1}{2}\log_2 n$, but there exist n-vertex graphs with no clique or independent set of size $2\log_2 n$. A graph that has no clique or independent set of size $C \cdot \log_2 n$ is called a C-Ramsey graph – although we know that 2-Ramsey graphs of every size must exist, we know of no explicit constructions of C-Ramsey graph families for any C (and we would like to find them because of relationships to randomness extractors.

Erdos Szemeredi 1972

For every $C > 0$, there exists some $\epsilon > 0$ such that for every sufficiently large n, any C-Ramsey graph on n vertices has density at least ϵ and at most $1 - \epsilon$.

Proof: *This proof will reference binary entropy; see the definition below.* Let $0 < \varepsilon < \frac{1}{16}$ be such that $H(8\varepsilon) < \frac{1}{4C}$. Suppose for contradiction that a C-Ramsey graph on n vertices (where n is large enough to have $C \cdot \log_2 n \leq \frac{n}{8}$ and $n \geq 2^{12}$) has density less than ε. So, G has average degree less than εn, meaning that at most $\frac{n}{2}$ vertices have degree $\geq 2\varepsilon n$. Let U be the set of vertices of G with degree at most $2\varepsilon n$; $|U| \geq \frac{n}{2}$. Let $I \subseteq U$ be an independent set of U of maximum size – because G was C-Ramsey, $|I| \leq C \cdot \log_2 n \leq \frac{n}{8}$. The number of edges from I to $U \setminus I$ must be at most $2\varepsilon n \cdot |I|$, since that's a bound on the total degree of vertices in I. So, at most $\frac{n}{4}$ of the vertices in $U \setminus I$ have at least $8\varepsilon|I|$ edges to I; let $W \subseteq U \setminus I$ be the vertices with at most than $8\varepsilon|I|$ edges to I. $|W| \geq |U| - |I| - \frac{n}{4} \geq \frac{n}{8}$. For every vertex $w \in W$, we choose $S(w) \subseteq I$ of size $s = \lfloor 8\varepsilon|I| \rfloor$ containing its neighbourhood in I.

The number of total subsets of I of size s is

$$\binom{|I|}{s} \leq 2^{H(s/|I|)\cdot|I|} \leq 2^{H(8\varepsilon)\cdot|I|} \leq 2^{\frac{1}{4C}\cdot C\log_2 n} = n^{\frac{1}{4}}$$

So, by pigeonhole principle, at least $\frac{n^{3/4}}{8}$ vertices in W have the same $S(w)$. Call those vertices W', and their shared neighbour set S. There are no edges from W' to $I \setminus S$. Because we assumed I was maximal, there can't be an independent set of size $s+1$ in W'. There also can't be a clique of size $t+1$, where $t = \lceil C \cdot \log_2 n \rceil - 1$ because G was C-Ramsey. So,

$$|W'| \leq R(s+1, t+1) \leq \binom{s+t}{s} \leq 2^{H(s/(s+t))\cdot(s+t)} \leq 2^{H(8\varepsilon)\cdot 2t} \leq 2^{\frac{1}{4C}\cdot 2C\log_2 n} \leq \sqrt{n} < \frac{n^{3/4}}{8} \leq |W'|$$

giving contradiction. ■

A brief review of binary entropy

For $0 \leq x \leq 1$, define the entropy of x as

$$H(x) = \begin{cases} 0 \text{ if } x = 0 \text{ or } x = 1; \text{ otherwise} \\ x\log_2\left(\frac{1}{x}\right) + (1-x)\log_2\left(\frac{1}{1-x}\right) \end{cases}$$

H is a continuous function on $[0,1]$ that's concave, symmetric, monotone increasing on $[0, \frac{1}{2}]$ and monotone decreasing on $[\frac{1}{2}, 1]$.
For any $0 \leq k \leq n$, we have

$$\binom{n}{k} \approx 2^{H(k/n)\cdot n}$$

Promel-Rodl 1999

For every $C > 0$, there exists a $\lambda > 0$ such that any C-Ramsey graph on n vertices contains every graph on $\lambda \log_2 n$ vertices as an induced subgraph.

Proof: We'll make use of a lemma very similar to one we've seen before.

Lemma: Let $0 \leq \rho \leq \frac{1}{2}$, $0 \leq \delta \leq \frac{1}{4}$, $s \geq \frac{2}{\delta}$. For every graph G, we have one of the following:

1. A vertex subset $U \subseteq V(G)$ of size $|U| \geq (\rho/2)^s|V(G)|$ such that all pairs of disjoint subsets $X, Y \subseteq U$ of size $|X| \geq \rho|U|$ and $|Y| \geq \rho|U|$, $1 - \delta \geq d(X,Y) \geq \delta$
2. There is a vertex subset $U \subseteq V(G)$ of size $|U| \leq V(G)$ of size $|U| \geq (\rho/2)^5 \cdot V(G)$ with density $d(U) \leq 3\delta$ or $d(U) \geq 1 - 3\delta$.

Proof: Left to the homework.
■

We can assume n is sufficiently large with respect to C. Let G be a C-Ramsey graph on n vertices. Note that every induced subgraph of G on at least $\sqrt{n}$ vertices is a $(2C)$-Ramsey graph. Let ε be as in the

Erdos-Szemeredi theorem for $2C$. Then, every induced subgraph of G on at least $\sqrt{n}$ vertices has density between ε and $1-\varepsilon$. Let $\delta = \frac{\varepsilon}{4}$, $s = \lceil \frac{2}{\delta} \rceil$, and $\rho = 2 \cdot n^{-1/(2s)}$. Option 2 of the lemma is impossible by our density bounds, so option 1 must hold.

13 Linear equation stuff

Consider a homogeneous system of linear equations with integer coefficients; i.e. $Ax = 0$ where $A \in \mathbb{Z}^{m\times k}$. We call this system **partition regular** if, for every integer t and any t-colouring of $\mathbb{N}$, there exists $x_1, \ldots, x_k$ of the same colour such that $A(x_1, \ldots x_k)^T = 0$.

Rado's Theorem

Let $A \in \mathbb{Z}^{m\times k}$ be an integer matrix, and let $a^{(1)}, \ldots, a^{(k)} \in \mathbb{Z}^m$ be its columns. Then $Ax = 0$ is a partition regular system iff: There exists a partition into nonempty subsets $I_0 \cup \cdots \cup I_l = [k]$ such that

$$\sum_{i\in I_0} a^{(i)} = 0$$

and for all j such that $1 \leq j \leq l$,

$$\sum_{i\in I_j} a^{(j)} \in \text{span}_{\mathbb{Q}}(a^{(h)} \mid h \in I_0 \cup \cdots \cup I_{j-1})$$

Proof: We'll first show that being partition regular implies the existence of a partition with the desired property. We will use the following lemma:

Lemma 1: Let $v_1, \ldots, v_l \in \mathbb{Z}^m$ and $v \in \mathbb{Z}^m$, $v \notin \text{span}_{\mathbb{Q}}(v_1, \ldots, v_l)$. There are only finitely many primes p such that $p^z v = \lambda_1 v_1 + \cdots + \lambda_l v_l \mod p^{z+1}$ for some integer $z \geq 0$ and integer coefficients $\lambda_1, \ldots, \lambda_l$.

Proof: Since $v \notin \text{span}_{\mathbb{Q}}(v_1, \ldots, v_l)$, we can find $w \in \mathbb{Z}^m$ such that $w \cdot v \neq 0$ but $w \cdot v_i = 0$ for $i \in \{1, \ldots, l\}$. Now, suppose we have

$$p^z v = \lambda_1 v_1 + \cdots + \lambda_l v_l \mod p^{z+1}$$

If we dot both sides by w, we get

$$p^z v \cdot w = 0 \mod p^{z+1}$$

Which implies that $p \mid w \cdot v$. Since $w \cdot v$ has only finitely many prime divisors, there are only finitely many options for p.

Suppose that $A \in \mathbb{Z}^{m\times k}$ is a matrix such that $Ax = 0$ is partition regular. Choose a prime p such that for any two subsets $I, J \subseteq [k]$ with $\sum_{j\in J} a^{(j)} \notin \text{span}_{\mathbb{Q}}(a^{(i)} | i \in I)$, we don't have

$$p^z \sum_{j\in J} a^{(j)} = \sum_{i\in I} \lambda_i a^{(i)} \mod p^{z+1}$$

for any $z \geq 0$ and $\lambda_i \in \mathbb{Z}$ for $i \in I$. The existence of such a prime follows from the lemma, since there's only finitely many choices for I and J. Now, consider a $p-1$ colouring of $\mathbb{N}$, where x is coloured by the value mod p of its largest factor not divisible by p. Since $Ax = 0$, we can find a solution to $Ax = 0$ such that $x_1, \ldots, x_k$ all have the same remainder r. Writing each x_i as $p^{z(x_i)} \cdot (p \cdot m(x_i) + r(x_i))$, we define $0 \leq z_0 < \cdots < z_l$ be all the values appearing among $z(x_1), \ldots, z(x_k)$. We partition into sets $I_j = \{h \mid z(x_h) = z_j\}$. We have

$Ax = 0$, so $x_1 a^{(1)} + \cdots + x_k a^{(k)} = 0$. For some specific $j \in \{0, \ldots, l\}$, note that taking this equation modulo p^{z_j+1} gives

$$x_1 a^{(1)} + \cdots + x_k a^{(k)} \mod p^{z_j+1} = \sum_{h \in I_0 \cup \cdots \cup I_{j-1}} x_h a^{(h)} + \sum_{h \in I_j} r p^{z_j} a^{(h)} \mod p^{z_j+1} = 0 \mod p^{z_j+1}$$

Multiplying by the modular inverse of r gives

$$p^{z_j} \sum_{h \in I_j} a^{(h)} \mod p^{z_j+1} = \sum_{h \in I_0 \cup \cdots \cup I_{j-1}} -r^{-1} x_h a^{(h)} \mod p^{z_j+1}$$

By our choice of p, this means that $\sum_{h \in I_j} a^{(h)}$ must be in the span of $a^{(h)} \mid h \in I_0 \cup \cdots \cup I_{j-1}$.

Now we'll show the reverse direction.

For integers $l, d, c > 0$, a subset $M \subseteq \mathbb{N}$ consisting for a fixed $y_0, \ldots, y_l \in \mathbb{N}$ of all integers of the form $cy_j + \lambda_{j+1} y_{j+1} + \cdots + \lambda_l y_l$ with $i \in \{0, \ldots, l\}$ and $\lambda_{j+1}, \ldots, \lambda_l \in \mathbb{Z} \cap [-d, d]$ is called an (l, d, c)-**Deuber set**.

Lemma 2: If $A \in \mathbb{Z}^{m \times k}$ has columns $a^{(1)}, \ldots, a(k)$ satisfying the condition of the theorem, there exist l, d, c such that every (l, d, c)-Deuber set $M \subseteq \mathbb{N}$ contains a solution to $Ax = 0$ with $x \in M^k$.

Proof: We assume that there exists a partition into nonempty subsets $I_0 \cup \cdots \cup I_l = [k]$ such that

$$\sum_{i \in I_0} a^{(i)} = 0$$

and for all j such that $1 \leq j \leq l$,

$$\sum_{i \in I_j} a^{(j)} \in \text{span}_{\mathbb{Q}}(a^{(h)} \mid h \in I_0 \cup \cdots \cup I_{j-1})$$

By taking common multiples of the denominators, we can find an integer $c > 0$ such that for all $j \in \{1, \ldots, l\}$

$$c \cdot \sum_{i \in I_j} a^{(i)} = \sum_{h \in I_0 \cup \cdots \cup I_{j-1}} \lambda_{j,h} a^{(h)}$$

with $\lambda_{j,h} \in \mathbb{Z}$. Define

$$d = \max_{j \in \{1, \ldots, l\}, h \in I_0 \cup \cdots \cup I_{j-1}} |\lambda_{j,h}|$$

Now, we claim that if M is an (l, d, c)-Deuber set for some $\{y_1, \ldots, y_l\}$, M contains a solution to $Ax = 0$.

For every $j \in \{0, \ldots, l\}$ and $i \in I_j$, define $x_i = cy_j - \lambda_{j+1,i} y_{j+1} - \cdots - \lambda l, i y_l$. Now,

$$\sum_{i=1}^{k} x_i a^{(i)} = \sum_{j=0}^{l} \sum_{i \in I_j} x_i a^{(i)} = \sum_{j=0}^{l} \sum_{i \in I_j} (cy_j - \lambda_{j+1,i} y_{j+1} - \cdots - \lambda l, i y_l) \, a^{(i)} =$$

$$\sum_{j=0}^{l} \left(\sum_{i \in I_j} c y_j a^{(i)} - \sum_{i \in I_0 \cup \cdots \cup I_{j-1}} \lambda j, i y_j a^{(i)} \right) =$$

$$\sum_{j=0}^{l} y_j \left(\sum_{i \in I_j} c a^(i) - \sum_{i \in I_0 \cup \cdots \cup I_{j-1}} \lambda j, i a^{(i)} \right) = 0$$

To prove the reverse direction of Rado's theorem, we show that for any l, d, c, t, any t-colouring of $\mathbb{N}$ has a monochromatic (l, d, c)-Deuber set. We will actually prove a stronger result due to Deuber: for any l, d, c, t, there exist l^*, d^*, c^* such that any t-coloured (l^*, d^*, c^*)-Deuber set contains a monochromatic (l, d, c)-Deuber set as a subset. Notably, this will give the following stronger version of Rado's theorem:

Deuber 1973

Let $M \in \mathbb{N}$ be a set containing solutions to all partition-regular systems $Ax = 0$. Then, for every colouring of M with finitely many colours, there exists a colour class that *also* contains solutions to all partition-regular systems.

To see the proof of the strengthening from the original theorem, note that there must be a colour class containing $(z, z \cdot z!, z!)$-Deuber sets for infinitely many z. This colour class must contain all (l, d, c) Deuber set.

Now, we return to the original theorem. We'll show by induction on h that for any $l, d, c, t > 0$, there exist l^*, d^*, c^* such that for every (l^*, d^*, c^*)-Deuber set $M^* \subseteq \mathbb{N}$ and every colouring of M^* with t colours, there exists an (l, d, c)-Deuber set $M \subseteq M^*$ such that for each $j \in \{0, \ldots, h\}$, the set $\{cy_j + \lambda_{j+1}y_{j+1} + \cdots + \lambda_j y_j \mid \lambda_{j+1}, \ldots, \lambda_l \in \mathbb{Z} \cap [-d, d]\}$ is monochromatic.

This result will apply our original theorem, because setting $t' = lt$ and letting $h = t'$, the pigeonhole principle will let us find l indices j where these sets have the same colour, which gives a monochromatic (l, d, c)-Deuber set.

WLOG $d \geq c$. For our base case, let $h = 0$ and apply the multi-dimensional (see pset) Hales-Jewett theorem with $S = \mathbb{Z} \cap [-d, d]$ and $m = l$. This gives some $n \in \mathbb{N}$ such that we can find a monochromatic l-parameter subset of $(\mathbb{Z} \cap [-d, d])^n$. Let $l^* = n$, $d^* = cd$, $c^* = c^2$, and let $M^*_{d^*,c^*}(y_0, \ldots, y_n)$ be an (l^*, d^*, c^*)-Deuber set with an associated t-colouring. We can use this to define a colouring of $S^n = (\mathbb{Z} \cap [-d, d])^n$ with t colours, by colouring $(\lambda_1, \ldots, \lambda_n) \in S^n$ by the colour of $c^2 y_0 + \lambda_1 c y_1 + \cdots + \lambda_n c y_n \in M^*$. By choice of n, there exists a monochromatic l-parameter subset in this colouring. Then, let x_0 be the sum of all the fixed terms in our l-parameter set, and x_j be the sum of coefficients for all appearances of the jth kind of $*$.

For our inductive case, take $h > 0$ and assume the statement for $h - 1$. Apply Hales-Jewett theorem with $S = \mathbb{Z} \cap [-d, d]$ and $m = l - h$, yielding $n \in \mathbb{N}$. Apply the induction assumption to $l' = n + h$, $d' = cd^2$, $c' = c^2$, $t' = t$, yielding l^*, d^*, c^*, and consider an (l^*, d^*, c^*)-Deuber set M^* with an associated t-colouring. Then, there exists an $(n + h, cd^2, c^2)$-Deuber set $M' \subseteq M^*$ such that for $j \in \{0, \ldots, h - 1\}$, $\{c^2 y_j + \lambda_{j+1} y_{j+1} + \cdots + \lambda_{n+h} y_{n+h} | \lambda_i \in \mathbb{Z} \cap [-cd^2, cd^2]\}$. Consider giving $(\lambda_{h+1}, \ldots, \lambda_{h+n}) \in S^n$ the colour of $c^2 y_h + \lambda_{h+1} c y_{h+1} + \cdots + \lambda_{h+n} c y_{h+n}$. We can find a monochromatic $(l - h)$-parameter subset by the Hales-Jewett theorem. Proceed as before.

■

14 Fancy new upper bounds

Recall that we know $\sqrt{2}^k \leq R(k, k,) \leq 4^k$. More generally, we know that $R(k, l) \leq \binom{k+l-2}{k-1}$. Here's an alternative "algorithmic" proof:

Claim: $R(k, l) \leq e \cdot (k + l) \cdot \frac{(k+l)^{k+l}}{k^k l^l}$.
Proof: Consider a red-blue-coloured complete graph G on n vertices, with no red clique of size k and no blue clique of size l. Initialize $X = V(G)$, $A = \{\}$, $B = \{\}$ and repeat the following until $|X| \leq k + l$:

1. Pick a vertex $x \in X$. Move x to A if more than a $\frac{k}{k+l}$ fraction of its neighbours in X are red, and to B otherwise.

2. If you've moved it to A, remove all of its blue neighbours from X, otherwise remove all its red

neighbours from X

We always have $|X| \geq \left(\frac{k}{k+l}\right)^{|A|} \cdot \left(\frac{l}{k+l}\right)^{|}B| \cdot \left(1-\frac{1}{k+l}\right)^{|A|+|B|} \cdot n$ Since there's no red k-clique or blue l-clique, we know $|A| < k$, $|B| < l$, and when the algorithm terminates we know $|X| \leq k+l$. Plugging into the above gives the desired lower bound on n.

This framing of the method turns out to be a useful approach to do better.

Campos Griffiths Morris Sahasrabudhe 2023

There exists an absolute constant $c > 0$ such that

$$R(k,k) \leq (4-c)^k$$

for all $k \geq 2$.

Proof: In a complete graph G with edges coloured red and blue, we say disjoint vertex sets $A, Y \subseteq G$ form a "red book" if A is a red clique and all edges between A and Y are red. Let G be a complete graph with edges coloured red and blue. Observe that if (A, Y) is a red book in G with $|Y| \geq R(k-|A|, l)$, then G contains a red clique of size k or a blue clique of size l.

Let $l \leq k$ and let G be a red-blue coloured complete graph with no red k-clique or blue l-clique. We can assume $l \geq \frac{k}{10}$. Our goal is again to upper bound $|V(G)|$. We do so by building disjoint subsets $A, B, X, Y \subseteq V(G)$ such that

- A is a red clique and all edges from A to X and to Y are red.
- B is a blue clique and all edges from B to X are blue.

We will track $|A|, |B|, |X|, |Y|$ and $p = d_{\text{red}}(X, Y)$. We define the "weight" of a pair $u, v \in X$ as $\omega(u,v) = \frac{1}{|Y|}(|N_{\text{red}}(u) \cap N_{\text{red}}(v) \cap Y) - p \cdot |N_(x) \cap Y|$, and define $\omega(x) = \sum_{v \in X \setminus \{x\}} \omega(x, v)$. If $\omega(x)$ is non-negative or just below 0, then $d_{\text{red}}(X \setminus \{x\}, N_{\text{red}}(x) \cap Y)$ is at least p or just below. Such a vertex will exist because $\sum_{u,v \in X} \omega(u,v) \geq 0$.

First, initialize $A = \{\}, B = \{\}$ and partition the vertices $V(G)$ into X and Y each of size $\frac{|V(G)|}{2}$ such that the $d_{\text{red}}(X, Y)$ is as large as the total red density. Let p_0 be the initial red density between X and Y. Let $\epsilon = k^{-1/4}$ and $q_h = p_0 + \frac{(1+\epsilon)^h - 1}{k}$, with $\alpha_h = q_h - q_{h-1}$. For $0 < p < 1$, define $h(p)$ to be the smallest $h \in \mathbb{N}$ such that $p \leq q_h$. Finally, we define $\mu = \frac{l}{k+l}$ for the case where $l < k$ and $\frac{2}{5}$ if $l = k$.

We repeat the following until $|X| \leq R(k, l^{3/4})$:

Remove all vertices $X \in X$ from X whose red neighbourhood in Y is significantly smaller than $p \cdot |Y|$. Then, take the first possible option of the following:

1. Big Blue Step: If there exist at least $R(k, \lceil l^{2/3} \rceil)$ vertices with blue neighbourhood of size $\geq \mu|X|$ in X, we can perform a big blue step. Find a blue book (S, T) in X with $\mu^{|S|} \cdot \frac{|X|}{2}$ for the largest $|S|$ possible. Move S to B, and set X to be T. This ensures $|S| \geq l^{1/4}$, so there are at most $l^{3/4}$ big blue steps.

2. Red Step: If that step fails, choose a vertex $x \in X$ with maximal $\omega(x)$ such that $|N_{\text{blue}}(x) \cap X| \leq \mu|X|$. If $d_{\text{red}}(N_{\text{red}}(x) \cap X, N_{\text{red}}(x) \cap Y) \geq p - \alpha_{h(p)}$, we can perform a red step; otherwise, we must perform a density-boost step. To take a red step, move x to A, and then intersect X and Y with the red neighbours of x.

3. Density-Boost Step: Add x to B, and intersect X with its blue neighbourhood and Y with its red neighbourhood.

First, we show $R(k,l) \leq e^{-l/50+o(k)} \cdot \binom{k+l}{l}$ for $\frac{k}{10} \leq l \leq \frac{k}{4}$. Then, we use this to show $R(k,k) \leq (4-c)^k$. Both proofs follow from the same algorithm with slightly different parameters. ■