The Preliminaries: NW Polynomial and Relative Rank

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A RECAP OF THE STORY

A recap of the story

- Set-multilinear polynomial: variable sets $X, Y, Z, \ldots$ – each monomial exactly one variable from each set (eg. $x_2y_1z_4 + 5x_3y_2z_2 - 12x_2y_2z_2$)

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- Counting argument $\rightarrow$ most set-multilinear polynomials don't have small depth-d set-multilinear formulas
- Goal: find such a polynomial in VNP – i.e. where coefficients are efficiently computable

Nisan-Wigderson Polynomial

$$NW_{n,d}(X) = \sum_{\substack{f(z) \in \mathbb{F}_n[z] \\ \deg(f) < d/2}} \prod_{j \in [d]} x_{f(j),j}$$

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monomial takes the $f(j)$th
variable from each X_j

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- So, any two monomials share fewer than $d/2$ variables

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$0j + 0$	$x_0 y_0 z_0 +$
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$\ldots$	$\ldots$
$3j + 4$	$x_4 y_7 z_2 +$

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$0j + 1$	$x_1y_1z_1+$
$\dots$	$\dots$
$3j + 4$	$x_4y_7z_2+$
$\dots$	$\dots$
$7j + 7$	$x_7y_6z_5$

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  - rows indexed by monomials on positive vars
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- relrk(p) with respect to the variable-set division :=

$$\frac{\text{rank of this matrix}}{\sqrt{\prod_i |X_i|}} = \frac{\text{rank of this matrix}}{\sqrt{n^d}}$$

Relative Rank Example: $n = 2, d = 4$

$$p = w_0x_0y_0z_0 + 4w_0x_1y_0z_1 + 4w_1x_0y_1z_0 + w_1x_1y_1z_1 + w_0x_0y_1z_1 + w_1x_1y_0z_0$$

	y_0z_0	y_0z_1	y_1z_0	y_1z_1
w_0x_0	1	0	0	1
w_0x_1	0	4	0	0
w_1x_0	0	0	4	0
w_1x_1	1	0	0	1

$$\frac{3}{\sqrt{2^4}} = \frac{3}{4}$$

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$$\begin{array}{c} \\ w_0z_0 \\ w_0z_1 \\ w_1z_0 \\ w_1z_1 \end{array} \begin{array}{c} \begin{array}{cccc} x_0y_0 & x_0y_1 & x_1y_0 & x_1y_1 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\frac{4}{\sqrt{2^4}} = 1$$

Relative Rank Example: $n = 2, d = 4$

$$p = w_0x_0y_0z_0 + 4w_0x_1y_0z_1 + 4w_1x_0y_1z_0 + w_1x_1y_1z_1 + w_0x_0y_1z_1 + w_1x_1y_0z_0$$

	$x_0y_0z_0$	$x_0y_0z_1$	$x_0y_1z_0$	$x_0y_1z_1$	$x_1y_0z_0$	$x_1y_0z_1$	$x_1y_1z_0$	$x_1y_1z_1$
w_0	1	0	0	1	0	4	0	0
w_1	0	0	4	0	1	0	0	1

$$\frac{2}{\sqrt{2^4}} = \frac{1}{2}$$

Facts about relrk: Normalization constant

Normalization constant $\frac{1}{\sqrt{n^d}}$ is such that relrk = 1 for a square matrix with full rank

FACTS ABOUT RELRK: NORMALIZATION CONSTANT

Normalization constant $\frac{1}{\sqrt{n^d}}$ is such that relrk = 1 for a square matrix with full rank

- Square matrix has half of the variable sets as rows and half as columns, so is $n^{d/2} \times n^{d/2}$

FACTS ABOUT RELRK: IMBALANCE

If the division of variable sets is unbalanced (i.e. there are t more positive sets than negative sets), then relrk is at most

$$\frac{n^{(d-t)/2}}{n^{d/2}} = n^{-t/2}$$

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- A matrix with $n^{(d-t)/2}$ rows and $n^{(d+t)/2}$ columns (or vice versa) has rank at most $n^{(d-t)/2}$.

Facts about relrk: Subadditivity

For a given vertex division,

$$\mathrm{relrk}(f+g) \leq \mathrm{relrk}(f) + \mathrm{relrk}(g)$$

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- Rank is subadditive

Facts about relrk: Multiplicativity

Suppose $p = fg$ where f is a set-multilinear polynomial over some of the variable sets, and g is a set-multilinear polynomial over the others. Then, relative to any division of the variable sets,

$$\text{relrk}(p) = \text{relrk}(f) \cdot \text{relrk}(g)$$

Facts about relrk: Multiplicativity

$$p = fg = (x_0y_0 + x_0y_1 + 3x_1y_1)(w_0z_0 + w_0z_1 + 2w_1z_0 + 2w_1z_1)$$

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$$\mathcal{M}(f) = \begin{array}{c|cc} & y_0 & y_1 \\ \hline x_0 & 1 & 1 \\ x_1 & 0 & 3 \end{array} \qquad \mathcal{M}(g) = \begin{array}{c|cc} & z_0 & z_1 \\ \hline w_0 & 1 & 1 \\ w_1 & 2 & 2 \end{array}$$

FACTS ABOUT RELRK: MULTIPLICATIVITY

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$$\mathrm{relrk}(p) = \prod_i \mathrm{relrk}(f_i)$$

relrk(NW) = 1 for all balanced divisions

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- There's exactly one degree-$< d/2$ polynomial that picks out those variables.
- So, each row has exactly one non-zero entry.
- Same logic shows each column has exactly one non-zero entry, so $\implies$ permutation matrix $\implies$ full rank

Brief history of partial derivative measures

60-second history lesson!

Notation pain

Ok, now we should probably look at the text of the paper and figure out what all their letters mean.