The Preliminaries: NW Polynomial and Relative Rank

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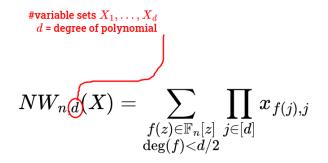
Set-multilinear polynomial: variable sets X, Y, Z, ... – each monomial exactly one variable from each set (eg. x₂y₁z₄ + 5x₃y₂z₂ - 12x₂y₂z₂)

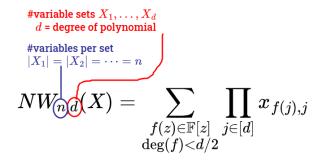
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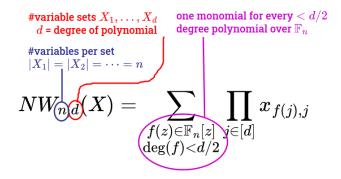
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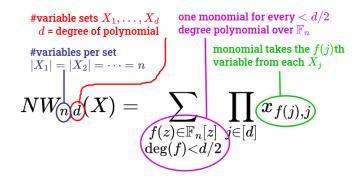
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- Goal: find such a polynomial in VNP i.e. where coefficients are efficiently computable

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The *d* variables involved in each monomial of NW_{n,d} correspond to evaluating a degree-*d*/2 polynomial on *d* points

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- The *d* variables involved in each monomial of NW_{n,d} correspond to evaluating a degree-< d/2 polynomial on *d* points
- So, any two monomials share fewer than d/2 variables

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0j + 1	$x_1y_1z_1 +$
3j + 4	$x_4y_7z_2 +$

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7 <i>j</i> + 7	<i>x</i> ₇ <i>y</i> ₆ <i>z</i> ₅

▶ Given variable sets X₁,..., X_d, divide the sets into "positive" and "negative" (e.g. X₁, X₂, X₃, X₄)

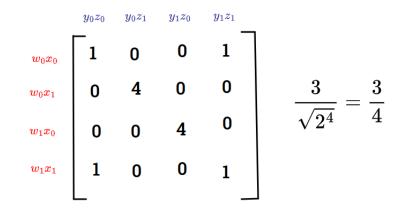
- ► Given variable sets X₁,..., X_d, divide the sets into "positive" and "negative" (e.g. X₁, X₂, X₃, X₄)
- For set-multilinear polynomial *p* over these vars, define "partial derivative matrix":
 - rows indexed by monomials on positive vars
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 - entry in a given (row, column) pair is the coefficient on the product of those monomials in p

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- relrk(p) with respect to the variable-set division :=

 $\frac{\operatorname{rank} \text{ of this matrix}}{\sqrt{\prod_i |X_i|}} = \frac{\operatorname{rank} \text{ of this matrix}}{\sqrt{n^d}}$

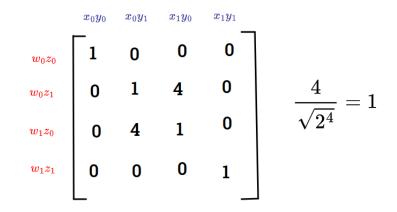
Relative Rank Example: n = 2, d = 4

 $p = w_0 x_0 y_0 z_0 + 4w_0 x_1 y_0 z_1 + 4w_1 x_0 y_1 z_0 + w_1 x_1 y_1 z_1 + w_0 x_0 y_1 z_1 + w_1 x_1 y_0 z_0$



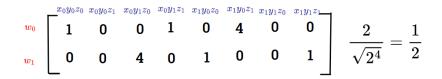
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FACTS ABOUT RELRK: NORMALIZATION CONSTANT

Normalization constant $\frac{1}{\sqrt{n^d}}$ is such that relrk = 1 for a square matrix with full rank

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 Square matrix has half of the variable sets as rows and half as columns, so is n^{d/2} × n^{d/2}

FACTS ABOUT RELRK: IMBALANCE

If the division of variable sets is unbalanced (i.e. there are *t* more positive sets than negative sets), then relrk is at most

$$\frac{n^{(d-t)/2}}{n^{d/2}} = n^{-t/2}$$

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► A matrix with n^{(d-t)/2} rows and n^{(d+t)/2} columns (or vice versa) has rank at most n^{(d-t)/2}.

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For a given vertex division,

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Rank is subadditive

Suppose p = fg where f is a set-multilinear polynomial over some of the variable sets, and g is a set-multilinear polynomial over the others. Then, relative to any division of the variable sets,

 $\operatorname{relrk}(p) = \operatorname{relrk}(f) \cdot \operatorname{relrk}(g)$

$$\mathcal{M}(f) = egin{bmatrix} y_0 & y_1 \ 1 & 1 \ 0 & 3 \end{bmatrix} \quad \mathcal{M}(g) = egin{matrix} z_0 & z_1 \ x_0 \ z_1 \end{bmatrix}$$

$$\mathcal{M}(f) \stackrel{x_0}{=} \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \qquad \mathcal{M}(g) \stackrel{y_1}{=} \begin{bmatrix} z_0 & z_1 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \qquad \mathcal{M}(g) \stackrel{w_0}{=} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$
$$\begin{array}{c} w_0 x_0 \\ w_1 x_0 \\ w_1 x_0 \\ w_1 x_0 \\ w_1 x_1 \\ w_1 x_1 \end{bmatrix} \begin{array}{c} 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

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$$\mathcal{M}(g) \stackrel{w_0 x_0}{=} \begin{bmatrix} y_0 z_0 & y_0 z_1 & y_1 z_0 & y_1 z_1 \\ w_1 x_0 \\ \mathcal{M}(g) & \mathcal{M}(g) \\ \mathcal{M}(g) = \\ w_0 x_1 \\ w_1 x_1 \end{bmatrix} \quad \begin{pmatrix} \mathcal{M}(g) & \mathcal{M}(g) \\ 0 & 3 \mathcal{M}(g) \\ & & \\ \end{pmatrix}$$

Suppose $p = f_1 \cdot \ldots \cdot f_j$, where each f_i is a set-multilinear polynomial over some of the variable sets (non-overlapping). Then, relative to any division of the variable sets,

$$\operatorname{relrk}(p) = \prod_{i} \operatorname{relrk}(f_i)$$

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- ► Same logic shows each column has exactly one non-zero entry, so ⇒ permutation matrix ⇒ full rank

BRIEF HISTORY OF PARTIAL DERIVATIVE MEASURES

60-second history lesson!

NOTATION PAIN

Ok, now we should probably look at the text of the paper and figure out what all their letters mean.